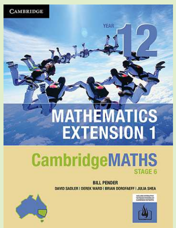
NORMANHURST BOYS HIGH SCHOOL

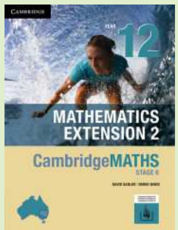
MATHEMATICS EXTENSION 1/2 (YEAR 12 COURSE)

Topic summary and exercises:

With references to

(X1) (X2) **Further Integration**





Name: ....

Initial version by H. Lam, 2011 (Volumes), October 2013 (Volume involving exponential & logarithmic functions) & February 2014 (Volume involving trigonometric functions, (X2) Further Integration Techniques).
Major revisions April 2020 for Mathematics Extension 1 & Extension 2. Last updated July 5, 2024.
Various corrections by students and members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under (cc) CC BY 2.0.

Symbols used

⚠ Beware! Heed warning.

(X1) Mathematics Extension 1 content.

(X2) Mathematics Extension 2 content.

Literacy: note new word/phrase.

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME12-4 uses calculus in the solution of applied problems, including differential equations and volumes of solids of revolution

MEX12-5 applies techniques of integration to structured and unstructured problems

Syllabus subtopics

ME-C2 Further Calculus Skills

ME-C3 Applications of Calculus

MEX-C1 Further Integration

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 12 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019) or *CambridgeMATHS Year 12 Extension 2* (Sadler & Ward, 2019) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

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Part I

(X1) (X2) Further Calculus Skills and Applications

Section 1

Integration by substitution

Learning Goal(s)

Knowledge
What is integration by substitution

Skills
Apply substitutions to evaluate integrals

Understanding
Why integration by substitution is needed

By the end of this section am I able to:

27.1 Find and evaluate indefinite and definite integrals using the method of integration by substitution, using a given substitution

1.1 Transforming functions

Fill in the spaces

- Most functions can be ....differentiated........ provided they are continuous and contain no sharp corners.
- However, it is far more difficult to find ....primitives........ .
- For some functions where a ....primitive........ cannot be found easily, it may be ....transformed........ into another function which is ....integrable........ .

Laws/Results

Diagram 1 A function $f(x)$ such that

- The ....primitive........ $F(x)$ cannot be found easily.
- Include an area for $x \in [a, b]$ which is to be found, i.e

$$\int_a^b f(x)\,dx$$

Diagram 2 A function $g(u)$ such that

- ....Transformed........ from $f(x)$
- The ....primitive........ $G(u)$ can be found.
- Include an equivalent area for $u \in [c, d]$ such that

$$\int_c^d g(u)\,du = \int_a^b f(x)\,dx$$

Important note

- Questions in the Extension 1 course *always* provide the required substitution.
  - Do *not* apply your own substitution.
- The substitution provided often reveals the ....chain.... ....rule.... ....residue.... in the integrand.

1.2 Indefinite integrals requiring a substitution

Example 1

Find $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ by using the substitution $u = \sqrt{x}$.

Solution

Steps

1. Replace all instances of $\sqrt{x}$ with u in the integrand:

2. Find $\dfrac{du}{dx}$ for $u = x^{\frac{1}{2}}$:

3. Separate the du and dx, to obtain a rule for dx in terms of du:

4. Replace dx with du:

5. Perform integration:

6. Transform back into x:

Example 2

[2019 Independent Ext 1 Trial Q7] What is an expression for $\int \frac{e^{2x}}{e^x+1}\,dx$ after substituting $u = e^x$?

(A) $\int \frac{u}{u+1}\,du$ (B) $\int \frac{2u}{u+1}\,du$ (C) $\int \frac{u^2}{u+1}\,du$ (D) $\int \frac{2u^2}{u+1}\,du$

Example 3

Find $\int \frac{x^2}{(x-2)^4}\,dx$ by using the substitution $x = u + 2$.

Example 4

Find $\int x\sqrt{1-x}\,dx$ using the substitution $u = 1 - x$. **Answer:** $\frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C$

Example 5

[1996 3U HSC Q1] (3 marks) Using the $u = e^x$, find $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$.

Example 6

[2018 JRAHS Ext 1 Trial Q14] (3 marks) Use the substitution $u = t^2 + 2$ to find

$$\int t^3\sqrt{t^2+2}\,dt$$

Answer: $\frac{1}{5}\left(t^2+2\right)^{\frac{5}{2}} - \frac{2}{3}\left(t^2+2\right)^{\frac{3}{2}} + C$

Example 7

[**2008 CSSA Ext 1 Trial Q1**] (3 marks) Using the substitution $u = \ln 3x$, find

$$\int \frac{dx}{x\left(\ln 3x\right)^2}$$

Answer: $-\frac{1}{\ln 3x} + C$

Example 8

[1998 3U HSC Q7/Ex 12D Q13]

i. Use the substitution $y = \sqrt{x}$ to find **3**

$$\int \frac{dx}{\sqrt{x(1-x)}}$$

ii. Use the substitution $z = x - \frac{1}{2}$ to find another expression for **3**

$$\int \frac{dx}{\sqrt{x(1-x)}}$$

iii. Use the results of parts (i) and (ii) to express $\sin^{-1}(2x-1)$ in terms of $\sin^{-1}\left(\sqrt{x}\right)$ for $0 < x < 1$. **1**

1.3 Definite integrals requiring a substitution

Important note

Remember to change the ...limits... of ...integration... .

Example 9

[2019 NBHS Ext 1 Trial Q12] Evaluate, by using the substitution $u^2 = 1 + x$:

$$\int_0^3 \frac{x(x+2)}{\sqrt{1+x}}\,dx$$

Answer: $\frac{52}{5}$

Example 10

[2019 Independent Ext 1 Trial Q12] Use the substitution $x = u^2$, $u \geq 0$ to evaluate in simplest exact form:

$$\int_1^{25} \frac{1}{2\left(x+\sqrt{x}\right)}\,dx$$

Answer: $\ln 3$

Example 11

[2019 CSSA Ext 1 Trial Q11] Use the substitution $u = \sqrt{x}$ to evaluate

$$\int_1^4 \frac{(\sqrt{x}-1)^3}{\sqrt{x}}\,dx$$

Answer: $\frac{1}{2}$

Example 12

[2019 Ext 1 HSC Q13/2017 Ext 2 HSC Q11] (3 marks) Use the substitution $u = \cos^2 x$ to evaluate

$$\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{4+\cos^2 x}\,dx$$

Answer: $\ln \frac{10}{9}$

Example 13

[2020 Ext 1 HSC Sample Q13] (3 marks) Using the substitution $x = \sin^2\theta$, or otherwise, evaluate

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx$$

Answer: $\frac{\pi}{4} - \frac{1}{2}$

Example 14

[2009 S&GPCA 3U Trial] (3 marks) Use the substitution $u = \tan^{-1} x$ to evaluate

$$\int_1^{\sqrt{3}} \frac{dx}{(1+x^2)\tan^{-1} x}$$

Answer: $\ln \frac{4}{3}$

Further exercises

Ex 12D (Pender et al., 2019)

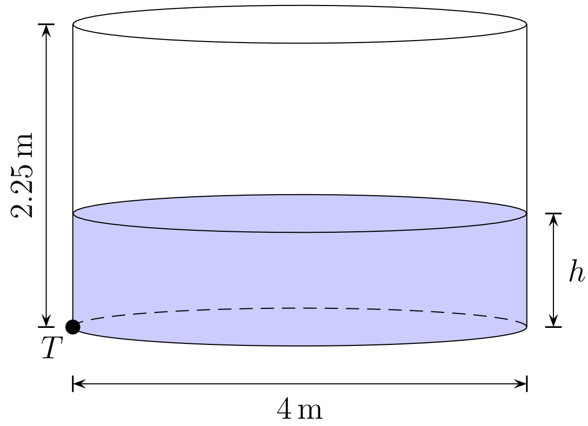
- Q2-12

Ex 12E (Pender et al., 2019)

- Q1-11

1.3.1 Supplementary exercises

1. **[2018 Baulkham Hills HS Ext 1 Trial Q13]** A cylindrical tank has diameter 4 m and height 2.25 m. Water is flowing into the tank at a rate of $\frac{2\pi}{5}$ m^3/min.



There is a tap at a point T at the base of the tank. When the tap is opened, water leaves the tank at a rate of $\frac{\pi}{5}\sqrt{h}$ m^3/min, where h is the height of the water in metres.

i. Show that at time t minutes after the tap has opened, the volume of water in the tank satisfies the differential equation **1**

$$\frac{dV}{dt} = \frac{\pi\left(2-\sqrt{h}\right)}{5}$$

ii. Show that at time t minutes after the tap has opened, the height of the water in the tank satisfies the differential equation **2**

$$\frac{dh}{dt} = \frac{2-\sqrt{h}}{20}$$

iii. When the tap is opened the height of the water is 0.16 metres. The time taken to fill the tank to a height of 2.25 metres can be calculated using **3**

$$t = \int_{0.16}^{2.25} \frac{20}{2-\sqrt{h}}\, dh \qquad \text{(Do NOT prove this)}$$

Using the substitution $h = (2-x)^2$, where $0 < x < 2$, find the time taken to fill the tank, correct to the nearest minute.

2. **[2017 Baulkham Hills HS Ext 1 Trial Q14]** - similar to 1996 Mathematics 4U HSC Q3

i. Using the substitution $u = \cos x$, show that, for any constant k, **2**

$$\int_0^{\frac{\pi}{2}} (\cos x)^{2k} \sin x \, dx = \frac{1}{2k+1}$$

ii. By noting that **2**

$$(\sin x)^{2n+1} = \sin x \left(1 - \cos^2 x\right)^n \qquad \text{(Do NOT prove this)}$$

show using the Binomial Theorem that, for all positive integers n:

$$\int_0^{\frac{\pi}{2}} (\sin x)^{2n+1} \, dx = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \left(\frac{1}{2r+1}\right)$$

iii. Use the result from part (ii) to evaluate **1**

$$\int_0^{\frac{\pi}{2}} (\sin x)^5 \, dx$$

Answers

1. 49 min **2.** $\frac{8}{15}$

Section 2

Volumes of solids of revolution

Learning Goal(s)

Knowledge
How to calculate volumes of solids of revolution by integration

Skills
Identifying whether to integrate along the x or the y axis

Understanding
Where the formula $\pi \int_a^b y^2\,dx$ or $\pi \int_a^b x^2\,dy$ arises from

By the end of this section am I able to:

27.2 Calculate area of regions between curves determined by functions

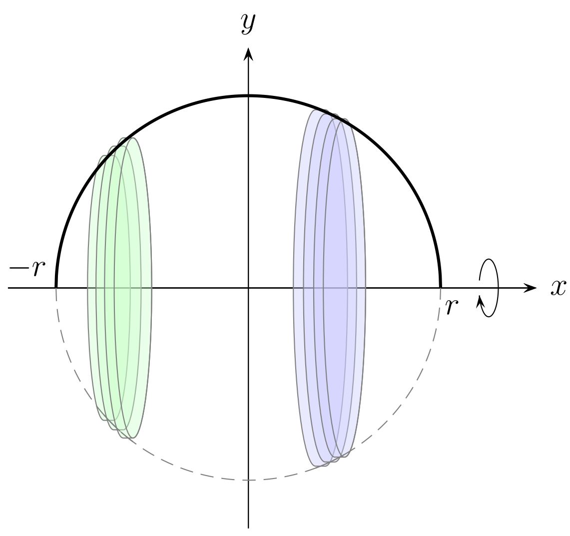
27.3 Sketch, with and without the use of technology, the graph of a solid of revolution whose boundary is formed by rotating an arc of a function about the x axis or y axis

27.4 Calculate the volume of a solid of revolution formed by rotating a region in the plane about the x axis or y axis, with and without the use of technology

27.5 Determine the volumes of solids of revolution that are formed by rotating the region between two curves about either the x axis or y axis in both real-life and abstract contexts

2.1 Derivation of formulae

- Volume of a right prism: ...... $V = Ad$ ......
  - Cross sectional area
  - Depth
- Volume of solids with circular cross sections
  - Rotate section of curve about x or y axis.
  - Add all thin slices of area
    - Radii that change
    - Sum from lower limit to upper limit



Laws/Results

Volume by rotating about

- x axis: ...... $V = \pi \int_a^b y^2\,dx$ ......
- y axis: ...... $V = \pi \int_a^b x^2\,dy$ ......

Example 15

Derive the formula for the volume of a sphere – $V = \frac{4}{3}\pi r^3$.

Steps

1. Draw diagram – hemisphere of radius r.

2. Formula: ....$y = \sqrt{r^2 - x^2}$........ .

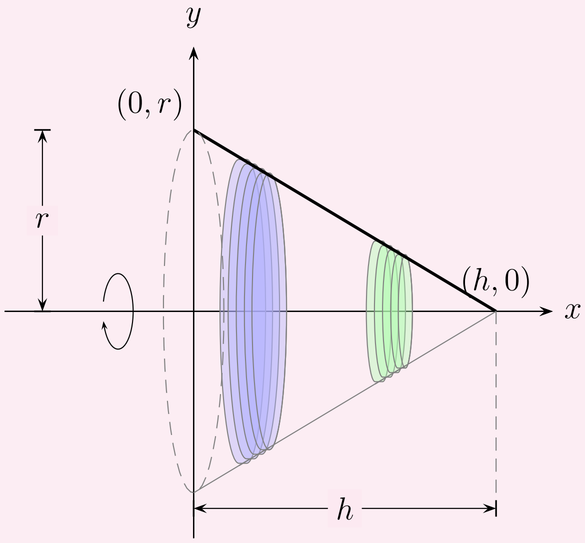
3. Apply volume formula:

Example 16

Derive the formula for the volume of a cone.

Steps

1. Recall the geometry of a cone:



2. Use two-point formula to find the equation of the line rotated about the x-axis:

3. Integrate from $x = 0$ to $x = h$:

2.2 Algebraic functions

Example 17

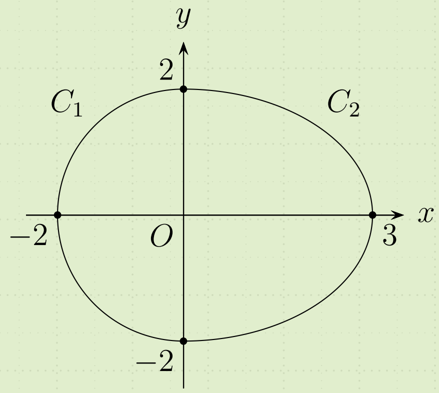
The area bounded by $y^2 = 3 - 2x - x^2$, $y \geq 0$ between $x = -3$ and $x = 1$ is rotated about the x axis.

Calculate the volume of the solid formed. **Answer:** $\frac{32\pi}{3}$

2.2.1 Addition of volumes

Example 18

[2016 2U HSC Q15] (4 marks) The diagram shows two curves C_1 and C_2. The curve C_1 is a semicircle $x^2 + y^2 = 4$, $-2 \leq x \leq 0$. The curve C_2 has equation $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, $0 \leq x \leq 3$.

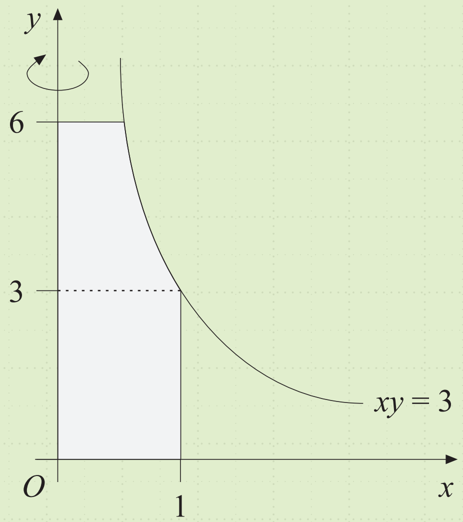


An egg is modelled by rotating the curves about the x axis to form a solid of revolution.

Find the exact value of the volume of the solid of revolution. **Answer:** $\frac{40\pi}{3}$

Example 19

[1995 3U HSC Q2] (4 marks) The shaded area is bounded by the curve $xy = 3$, the lines $x = 1$ and $y = 6$, and the two axes. A solid is formed by rotating the shaded area about the y axis.

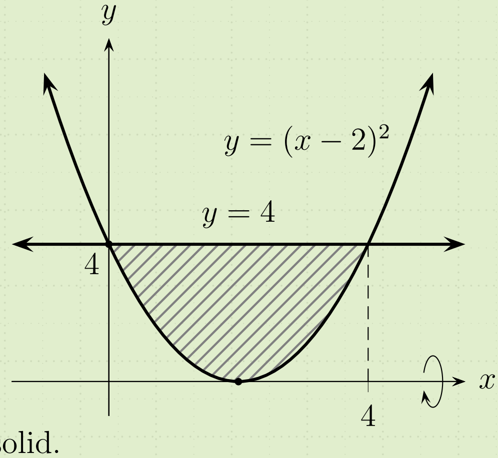


Find the volume of this solid by considering separately the regions above and below $y = 3$. **Answer:** $\frac{9\pi}{2}$

2.2.2 Subtraction of volumes

Example 20

[2010 CSSA 2U Trial Q7] (4 marks) The shaded region bounded by $y = (x-2)^2$ and $y = 4$ is rotated about the x axis to form a solid of revolution as shown.



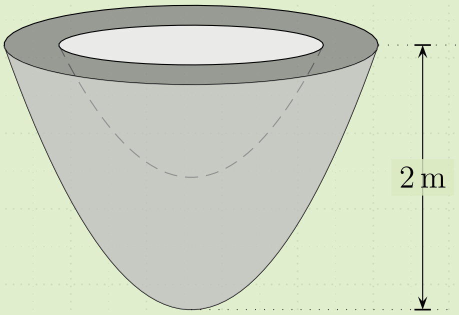
Find the volume of the solid. **Answer:** $\frac{256\pi}{5}$

Important note

⚠ Beware! Top volume subtract bottom volume, not top 'curve' subtract bottom 'curve'.

Example 21

[2021 Ext 1 HSC Q13] (3 marks) A 2-metre-high sculpture is to be made out of concrete. The sculpture is formed by rotating the region between $y = x^2$, $y = x^2 + 1$ and $y = 2$ around the y axis.



Find the volume of concrete needed to make the sculpture. **Answer:** $\frac{3\pi}{2}$

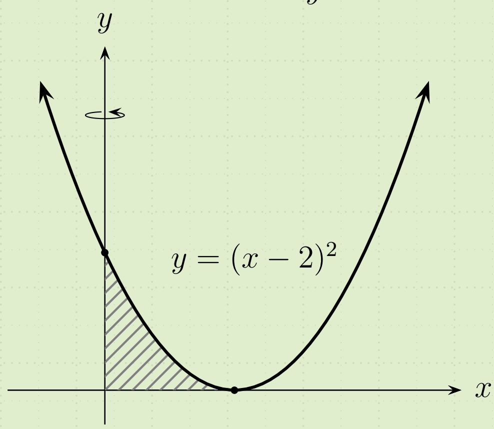
Important note

- Beware of the axis of integration and ...limits... of integration!
- ...Draw... ...picture...

2.2.3 Harder questions

Example 22

[2013 2U HSC Q15] (4 marks) ⚠ The region bounded by the x axis, y axis and the parabola $y = (x-2)^2$ is rotated about the y axis to form a solid.



Find the volume of the solid. **Answer:** $\frac{8\pi}{3}$

Example 23

[2017 Sydney Grammar Ext 1 Q12]

i. Use the substitution $u = 3x + 1$ to show that **2**

$$\int_0^1 \frac{x}{(3x+1)^2}\,dx = \frac{2}{9}\ln 2 - \frac{1}{12}$$

ii. Hence find the volume of the solid formed when the region bounded by the curve $y = \dfrac{6\sqrt{x}}{3x+1}$, the x axis and the line $x = 1$ is rotated about the x axis. Give your answer in exact form. **1**

Answer: $\pi\left(8\ln 2 - 3\right)$

2.2.4 Supplementary exercises

1. (Simple volumes) Calculate the volume of the solid formed when the regions bounded by the following curves are rotated about the given axis. First draw a diagram showing the region and the resulting solid.

(a) $y = 3$, x axis, $x = 1$, $x = 5$; about x axis

(b) $y = 2x$, x axis, $x = 4$; about x axis

(c) $x = y^2$, y axis, $y = 3$; about y axis

(d) $y = \sqrt{9 - x^2}$ and the x axis; about x axis

(e) $y = 2x - 8$, x axis, $x = 1$, $x = 3$; about x axis

(f) $x = \sqrt{y}$, y axis, $y = 1$, $y = 4$; about y axis

(g) $y = x^2 - 3x$ and the x axis; about x axis

(h) $y = -\frac{3}{2}x + 3$ and the coordinate axes, about the y axis

(i) $y = (2x - 1)^2$ and the coordinate axes;

i. about the x axis

ii. about the y axis

(j) $y = x + \frac{1}{x}$, the x axis, $x = \frac{1}{2}$, $x = 2$; about the x axis

2. (Compound volumes) Calculate the volume of the solid formed when the region bounded by the given curves is rotated about the given axis.

(a) $y = \sqrt{x}$, the y axis, and $y = 3$; about the x axis

(b) $y = \frac{x}{2}$, $x = y^2$

i. about the x axis

ii. about the y axis

(c) $y = x^2$ and $y = x^3$

i. around the x axis

ii. around the y axis

3. **[2010 NSBHS 2U Trial Q6]** (4 marks) Find the volume generated when the area between the curve $y = x^3$ and the line $y = x$ where $x \geq 0$ is rotated about the x axis.

4. (Trapezoidal rule) Use the trapezoidal rule to approximate the volume of the following solids:

(a) The region bounded by $y = \dfrac{1}{\sqrt{x}}$, the x axis, and the lines $x = 1$ and $x = 4$ is rotated about the x axis. (Use four function values)

(b) The region bounded by $y = 2^x$, the x axis, and the lines $x = 1$ and $x = 2$ is rotated about the x axis. (Use three function values)

Extension

5. Calculate the volume of the solid formed when the regions bounded by the following curves are rotated about the given axis.

(a) $y = \sqrt{9 - x^2}$, $y = 18 - 2x^2$; about x axis.

(b) $x^2 = 4y$, $y = (2x - 5)^2$, the x axis; about x axis.

(c) $y = x^2 - 1$ and $y = \frac{1}{2}x^2 + 1$ in the first quadrant, about the y axis.

6. (a) The region bounded by $y = \frac{1}{x}$, the x axis, and the lines $x = 1$ and $x = a$ (where $a > 1$) is rotated about the x axis. Find the volume of the resulting solid.

(b) Hence, show that when the whole of this curve to the right of $x = 1$ is rotated about the x axis, the volume of the resulting solid is finite, and find this volume.

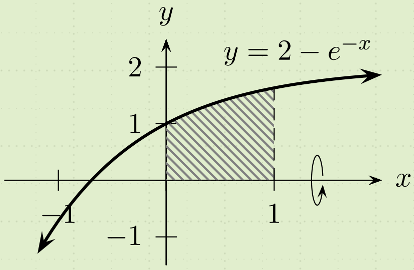
Answers to supplementary exercises §2.2.4 on the facing page

1. (a) 36π (b) $\frac{256\pi}{3}$ (c) $\frac{243\pi}{5}$ (d) 36π (e) $\frac{104\pi}{3}$ (f) $\frac{15\pi}{2}$ (g) $\frac{81\pi}{10}$ (h) 4π (i) i. $\frac{\pi}{10}$ ii. $\frac{\pi}{24}$ (j) $\frac{57\pi}{8}$ **2.** (a) $\frac{81\pi}{2}$ (b) i. $\frac{8\pi}{3}$ ii. $\frac{64\pi}{15}$ (c) i. $\frac{2\pi}{35}$ ii. $\frac{\pi}{10}$ **3.** $\frac{4\pi}{21}$ **4.** (a) $\frac{35\pi}{24} \approx 4.581$ (b) $18\pi \approx 56.5$ **5.** (a) $\frac{5\,004\pi}{5}$ (b) $\frac{272\pi}{81}$ (c) $\frac{7\pi}{2}$ **6.** (a) $\pi\left(1 - \frac{1}{a}\right)$ units^3 (b) π units^3

2.3 Exponential and logarithmic functions

Example 24

[2013 S&GPCA Trial] The region shaded is bounded by $y = 2 - e^{-x}$, the x axis, the y axis and the line $x = 1$.

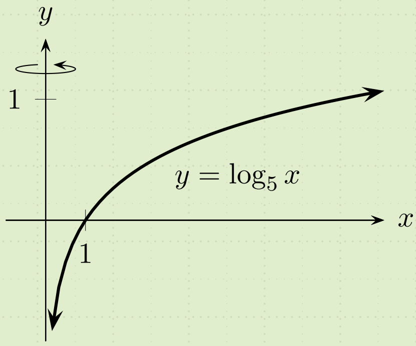


Find the volume of the solid formed when the shaded region is rotated about the x axis.

Answer: $\pi\left(\frac{e^2+8e-1}{2e^2}\right)$ units^3

Example 25

[2013 CSSA 2U Trial] The diagram shows the graph of $y = \log_5 x$. The region bounded by the line $y = \log_5 x$, the line $y = 1$ and the coordinate axes are rotated about the y axis to form a solid.



(i) Show that the volume of the solid is given by **3**

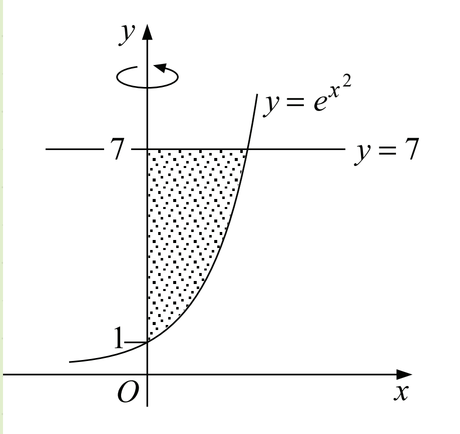
$$V = \pi \int_0^1 e^{y \log_e 25} \, dy$$

(ii) Hence find the volume of the solid. **2**

Answer: $\frac{24\pi}{\log_e 25}$

Example 26

[1999 2U HSC Q8] The shaded region bounded by $y = e^{x^2}$, $y = 7$ and the y axis is rotated about the y axis to form a solid.



i. Show that the volume of this solid is given by **2**

$$V = \pi \int_1^7 \log_e y \, dy$$

ii. Copy and complete the table. Give your answers correct to 3 decimal places. **2**

y	1	4	7
$\log_e y$			

iii. Use the trapezoidal rule with 3 function values to approximate the volume, V. **3**

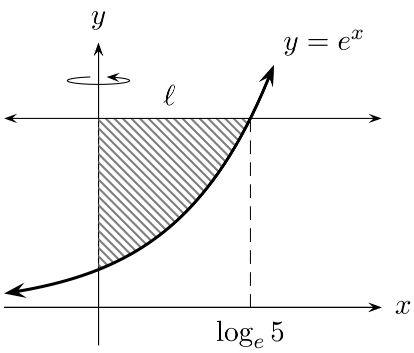
Example 27

Find the volume when the area enclosed by $y = \dfrac{x+1}{x}$, the x axis and the line $x = -5$ is rotated about the x axis. **Answer:** $\pi\left(\frac{24}{5} - 2\log_e 5\right)$

2.3.1 Additional questions

1. Find the volume formed when the area bounded by the curve $y = \dfrac{2}{\sqrt{3x+2}}$, the x axis, the lines $x = 0$ and $x = 2$ is rotated about the x axis.

2. **[2003 2U HSC Q8]** In the diagram, the shaded region is bounded by the y axis, the curve $y = e^x$ and a horizontal line ℓ that cuts the curve at a point whose x coordinate is $\log_e 5$.



A solid is formed by rotating the shaded region about the y axis.

Write down a definite integral whose value is the volume of the solid. (Do NOT evaluate the integral.)

Answers

1. $\frac{8\pi}{3}\log_e 2$ **2.** $\pi\int_1^5 (\ln y)^2\ dy$

2.4 Trigonometric Functions

Example 28

[2010 St George Girls' HS 2U Trial]

(i) Show that the volume of the solid formed when $y = \tan 2x$ is rotated about the x axis between $x = 0$ and $x = \frac{\pi}{6}$ is given by **2**

$$V = \pi \int_0^{\frac{\pi}{6}} \left(\sec^2 2x - 1\right)\, dx$$

(ii) Find the exact volume of the solid. **2**

Answer: $\pi\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$

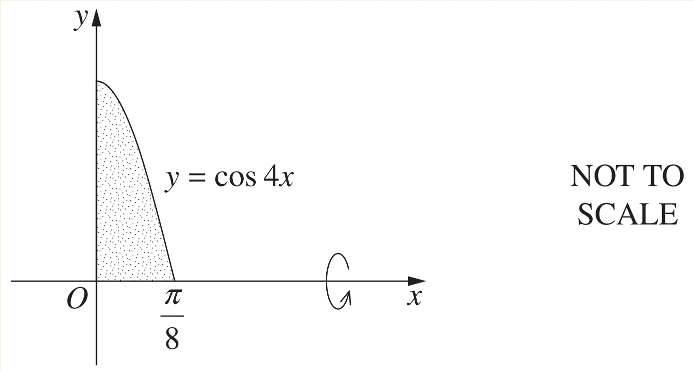
Example 29

[**2003 CSSA 2U Trial**] Find the volume generated when the curve $y = \sqrt{\cot x}$ is rotated about the x axis between $x = \frac{\pi}{3}$ and $x = \frac{\pi}{4}$. Leave your answer in exact form.

Answer: $\pi \ln \frac{\sqrt{6}}{2}$

Example 30

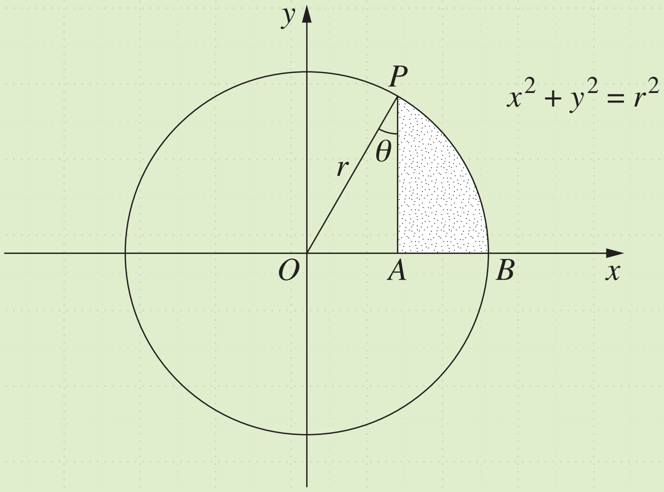
[2014 Ext 1 HSC Q12] (3 marks) The region bounded by $y = \cos 4x$ and the x axis, between $x = 0$ and $x = \dfrac{\pi}{8}$ is rotated about the x axis to form a solid.



Find the volume of the solid. **Answer:** $\frac{\pi^2}{16}$

Example 31

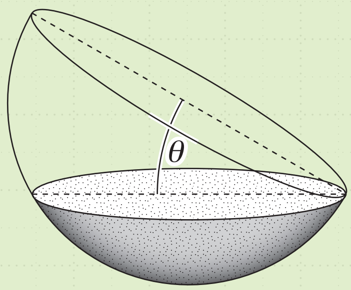
[2010 2U HSC Q10] The circle $x^2 + y^2 = r^2$ has radius r and centre O. The circle meets the positive x axis at B. The point A is on the interval OB. A vertical line through A meets the circle at P. Let $\theta = \angle OPA$.



i. The shaded region bounded by the arc PB and the intervals AB and AP is rotated about the x axis. Show that the volume, V, formed is given by **3**

$$V = \frac{\pi r^3}{3}\left(2 - 3\sin\theta + \sin^3\theta\right)$$

ii. A container is in the shape of a hemisphere of radius r metres. The container is initially horizontal and full of water. The container is then tilted at an angle of θ to the horizontal so that some water spills out.



(α) Find θ so that the depth of water remaining is one half of the original depth. **1**

(β) What fraction of the original volume is left in the container? **2**

Answer: $\theta = \frac{\pi}{6}$. $\frac{5}{16}$ of original volume left

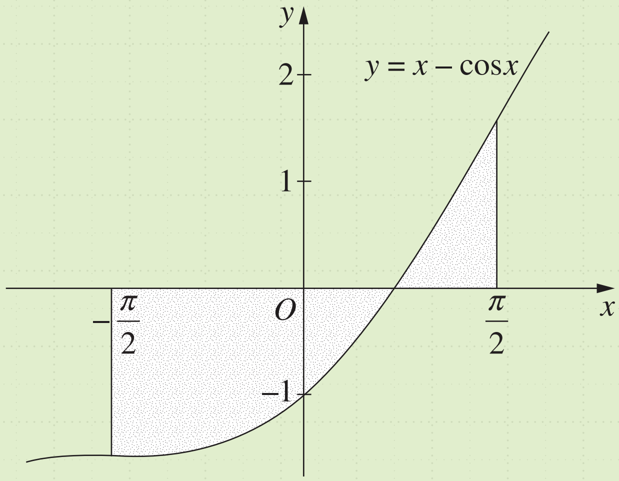
Example 32

[2020 Ext 1 HSC Sample Q14]

i. Sketch the graph of $y = x\cos x$ for $-\pi \le x \le \pi$ and hence explain why **3**

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\,dx = 0$$

ii. Consider the volume of the solid of revolution produced by rotated about the x axis the shaded region between the graph of $y = x - \cos x$, the x axis and the lines $x = -\dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2}$. **3**



Using the results of part (i), or otherwise, find the volume of the solid.

Answer: $\frac{\pi^4 + 6\pi^2}{12}$

2.5 Inverse Trigonometric Functions

Important note

- The primitives of $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ are *not* a part of the course.
- Use y axis to assist where necessary.

Example 33

[2007 Ext 1 HSC Q3] (3 marks) Find the volume of the solid of revolution formed when the region bounded by the curve $y = \dfrac{1}{\sqrt{9+x^2}}$, the x axis, the y axis and the line $x = 3$, is rotated about the x axis.

Example 34

[2006 CSSA Ext 1 Trial] (4 marks) The region in the first quadrant bounded by the curve $y = 2\tan^{-1} x$, the y axis, the lines $y = 0$ and $y = \frac{\pi}{2}$ is rotated through one complete revolution about the y axis.

Find the exact volume of the solid of revolution formed. **Answer:** $\frac{\pi}{2}(4 - \pi)$

Example 35

[2006 Independent Ext 1 Trial] Consider the function $y = \frac{1}{2}\cos^{-1}(x-1)$.

(i) Find the domain and range of this function. **2**

(ii) Sketch the graph of the function showing clearly the coordinates of the end-points. **1**

(iii) The region in the first quadrant bounded by the curve $y = \frac{1}{2}\cos^{-1}(x-1)$ and the coordinate axes is rotated through 360° about the y axis. Find the volume of the solid of revolution, giving your answer in simplest exact form. **3**

Answer: $\frac{3}{4}\pi^2$

Further exercises

Ex 12F (Pender et al., 2019)

- Q2-21

Part II

(X2) Further Integration Techniques

Section 3

Integrating rational functions

Learning Goal(s)

Knowledge
How to integrate rational functions

Skills
Transforming algebra to alter the appearance of a numerator or denominator

Understanding
Why such transformations are needed

By the end of this section am I able to:

27.6 Integrate rational functions involving a quadratic denominator by completing the square or otherwise

- These types can be easily transformed into Extension 1 integrals.

3.1 Quadratic with linear term in the denominator

- Denominator consists of quadratic (or function of a quadratic) in the form $ax^2 + bx + c$.
- Complete the square to obtain a transformed standard integral.

Example 36

Find $\displaystyle\int \frac{1}{\sqrt{3 + 2x - x^2}}\,dx$.

Answer: $\sin^{-1}\frac{x-1}{2} + C$

Example 37

Find the value of $\displaystyle\int_{-1}^{1} \frac{9}{7 + 4x + x^2}\,dx$.

Answer: $\frac{\pi\sqrt{3}}{2}$

3.2 Quadratic with linear term in the numerator

- Split numerator judiciously to obtain standard integrals.
- Aim for multiple of the derivative of the quadratic in the denominator, plus constant.

Example 38

Evaluate $\displaystyle\int \frac{4x+3}{x^2+9}\,dx$.

Answer: $2\ln\left(x^2+9\right)+\tan^{-1}\frac{x}{3}+C$

Example 39

Find $\displaystyle\int \frac{4x^3-2x^2+1}{2x-1}\,dx$.

Answer: $\frac{2}{3}x^3+\frac{1}{2}\ln(2x-1)+C$

3.3 Rationalising the numerator

- Make the numerator rational when a surd appears.

Example 40

[Ex 4E Q6] Evaluate $\int \sqrt{\frac{1+x}{1-x}}\,dx$.

Answer: $\sin^{-1} x - \sqrt{1-x^2} + C$

Further exercises

Ex 4A (Sadler & Ward, 2019)

- Q1-5

Ex 4B (Sadler & Ward, 2019)

- Q1-7

Ex 4E (Sadler & Ward, 2019)

- Q1(a)(b)
- Q2(a)-(d)
- Q3(a)-(d)

Section 4

Integration by substitution

Learning Goal(s)

Knowledge
What is integration by substitution

Skills
Apply appropriate substitutions to evaluate integrals

Understanding
How to develop the substitution

By the end of this section am I able to:

27.1 Find and evaluate indefinite and definite integrals using the method of integration by substitution, using a given substitution

- (x2) Substitution may not be given!

Important note

Look for the ....chain.... ....rule.... ....residue.... to create the substitution, where none is given.

Example 41

[2017 Girraween HS Ext 2 Trial HSC Q12] Find

$$\int \frac{\sqrt{x^2-1}}{x}\,dx$$

Answer: $\sqrt{x^2-1} - \cos^{-1}\frac{1}{x} + C$

Example 42

Use a suitable substitution to find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)^3}\,dx$. **Answer:** $\frac{3}{8}$

Example 43

Find $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$.

Answer: $\tan^{-1}\sqrt{e^{2x}-1}+C$

Further exercises

Ex 4C (Sadler & Ward, 2019)

- Q1-13

Section 5

Partial fractions

Learning Goal(s)

Knowledge
What are partial fractions

Skills
Decompose rational functions into its partial fractions

Understanding
Why decomposition into partial fractions is needed

By the end of this section am I able to:

27.7 Decompose rational functions whose denominators have simple linear or quadratic factors, or a combination of both, into partial fractions

5.1 Decomposition into partial fractions

5.1.1 Rationale

- Simplify $\dfrac{1}{x+1} + \dfrac{1}{x+2}$.

- Rewrite $\dfrac{2x+3}{(x+1)(x+2)}$ as the sum of two fractions.

5.1.2 Classifications of rational functions

Denominator with distinct linear factors

Example 44

Express $\dfrac{5}{(x+3)(2x+1)}$ as the sum of two partial fractions. **Answer:** $-\frac{1}{x+3}+\frac{2}{2x+1}$

Solution

Steps

1. Let $\dfrac{5}{(x+3)(2x+1)} =$ .... $\dfrac{A}{(x+3)} + \dfrac{B}{2x-1}$ ....

2. Multiply both sides by .... $(x-1)(x-2)$ .... :

3. Substitute "convenient" values to determine A and B:

Alternatively,

- Add fractions to obtain .... $\dfrac{A(2x+1)+B(x+3)}{(x+3)(2x+1)}$ ....
- Equate coefficients and solve:

Example 45

Express $\dfrac{3x-2}{(x-1)(x-2)}$ as a sum of partial fractions. **Answer:** $-\frac{1}{x-1}+\frac{4}{x-2}$

Example 46

Rewrite $\dfrac{x-1}{x^2-2x-3}$ as a sum of partial fractions. **Answer:** $\frac{1}{2(x+1)}+\frac{1}{2(x-3)}$

Example 47

Express $\dfrac{x^2+1}{(x-3)(x+2)}$ as a sum of partial fractions.

Answer: $1+\frac{2}{x-3}-\frac{1}{x+2}$

Denominator with unfactorisable quadratic factor over $\mathbb{R}$

Example 48

Express $\dfrac{4x+2}{(x+3)(x^2+1)}$ as a sum of partial fractions.

Answer: $-\frac{1}{x+3}+\frac{x+1}{x^2+1}$

Solution

Steps

1. Let $\dfrac{4x+2}{(x+3)(x^2+1)} = \dfrac{A}{x+3}+\dfrac{Bx+C}{x^2+1}$

2. Multiply both sides by $(x+3)(x^2+1)$

3. Substitute convenient values:

Example 49

Express $\dfrac{x^3+1}{(x^2+2)(x^2+8)}$ as a sum of partial fractions. **Answer:** $\frac{-2x+1}{6(x^2+2)} + \frac{8x-1}{6(x^2+8)}$

Example 50

Express $\dfrac{x^2+4x}{(x-1)(4x^2+1)}$ as a sum of partial fractions. **Answer:** $\frac{1}{x-1} + \frac{-3x+1}{4x^2+1}$

Denominator with distinct quadratic factors

Example 51

Express $\dfrac{x^2+6x+5}{(x-2)(x^2+x+1)}$ as the sum of partial fractions. **Answer:** $\frac{3}{x-2}-\frac{2x+1}{x^2+x+1}$

Example 52

Express $\dfrac{3x^2-2x+1}{(x^2+1)(x^2+2)}$ as the sum of partial fractions. **Answer:** $\frac{2x+5}{x^2+2}-\frac{2x+2}{x^2+1}$

Example 53

Express $\dfrac{54}{(x^2+x-20)(x-1)}$ as the sum of partial fractions. **Answer:** $\frac{1}{x+5}+\frac{2}{x-4}-\frac{3}{x-1}$

Further exercises

Ex 4D (Sadler & Ward, 2019)

- Q1

5.2 Integration

Learning Goal(s)

Knowledge
What are partial fractions

Skills
Integrate rational functions which have been decomposed into its partial fractions

Understanding
Why decomposition into partial fractions is needed

By the end of this section am I able to:
27.8 Use partial fractions to integrate functions

Example 54

Decompose $\dfrac{x+1}{(x-1)(x+3)}$ into partial fraction, and evaluate $\displaystyle\int_2^6 \frac{x+1}{(x-1)(x+3)}\,dx$.

Answer: $\ln 3$

Example 55

Show that $\dfrac{x^3 + x - 3}{x^2 - 3x + 2} = x + 3 + \dfrac{7}{x-2} + \dfrac{1}{x-1}$, and hence find its primitive.

Example 56

Rewrite $\dfrac{3x+10}{(x-2)(x^2+4)}$ into partial fractions, and determine its primitive.

Answer: $2\ln(x-2)-\ln\left(x^2+4\right)-\frac{1}{2}\tan^{-1}\frac{x}{2}+C$

Example 57

(a) Find A, B and $C \in \mathbb{R}$ such that

$$\frac{8-x}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}$$

(b) Hence evaluate $\displaystyle\int_0^1 \frac{8-x}{(x-2)^2(x+1)}\,dx$.

Answer: $A=-1$, $B=2$, $C=1$, $1+2\ln 2$

Example 58

Evaluate $\displaystyle\int \frac{x^2}{(x-2)(x+2)(x+3)}\,dx$

Answer: $\frac{1}{5}\ln(x-2) - \ln(x+2) + \frac{9}{5}\ln(x+3)$

Further exercises

Ex 4D (Sadler & Ward, 2019)

- Q2-13

Ex 4E (Sadler & Ward, 2019)

- Q4 onwards.

Important note

⚠ Beware of $\displaystyle\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx$ as these are not in the syllabus, and the result provided in Ex 4E will be supplied in an examination situation if it is ever required.

Further exercises (Legacy Textbooks)

Ex 2G (Patel, 2004)

Ex 4.4 (Lee, 2006)

Section 6

Integration by parts

Learning Goal(s)

Knowledge
How to integrate functions which are composed a product of two other functions

Skills
Identifying the u and dv terms

Understanding
Why integration by parts is the 'reverse rule' for product rule for differentiation

By the end of this section am I able to:

27.9 Evaluate integrals using the method of integration by parts

6.1 Derivation: rearrangement of the product rule result

Theorem 1

To integrate the product of two functions, choose carefully $u(x)$ (denoted u) and $v'(x)$ (denoted v'):

$$\int u\,dv = uv - \int v\,du \tag{9.1}$$

Proof

Steps

1. Write product rule, given $f(x) = u(x)v(x)$.

$$f'(x) = u(x)v'(x) + v(x)u'(x)$$

2. Rearrange,

$$u(x)v'(x) = f'(x) - v(x)u'(x)$$

3. Integrate both sides of equation,

Example 59

Use integration by parts to evaluate $\int xe^x \, dx$. **Answer:** $e^x(x-1)+C$

Steps

1. Let $u(x) = x$:

2. Write u, u' (du), v, v' (dv).

3. Evaluate integral, given formula.

Example 60

Evaluate $\int_0^{\pi} (x+1)\sin x \, dx$. **Answer:** $\pi + 2$

Steps

1. Let polynomial be $u(x)$:

2. Write u, du, v, dv.

3. Evaluate integral, given formula.

6.2 Repeated application

Example 61

Evaluate $\displaystyle\int_0^1 x^2 e^{-x}\, dx$. **Answer:** $2 - 5e^{-1}$

6.3 Exceptions with polynomials

Important note

Usually, the polynomial term is reduced in power, i.e. nominate the polynomial to be $u(x)$, but with logarithmic functions, this may present an exception.

Example 62

Evaluate $\displaystyle\int x \ln x \, dx$.

Answer: $\frac{1}{4}x^2 (2 \ln x - 1) + C$

Example 63

[2011 Independent] Evaluate in simplest exact form: $\displaystyle\int_1^e x^3 \ln x \, dx$.

Answer: $\frac{1}{16}(3e^4 + 1)$

6.4 Inserting phantom polynomial

⚠ Phantom polynomial: $P(x) = 1$.

Example 64

Evaluate $\displaystyle\int \sin^{-1} x \, dx$ **Answer:** $x \sin^{-1} x + \sqrt{1 - x^2} + C$

Example 65

Evaluate $\displaystyle\int \log_e x \, dx$ **Answer:** $x \ln x - x + C$

6.5 Recurrence of integral

- Assign variable to integral, usually I.

Example 66

Find the primitive of $e^x \sin x$.

Answer: $\frac{1}{2}e^x\left(\sin x - \cos x\right) + C$

Further exercises

Ex 4F (Sadler & Ward, 2019)

- Q1-14

Further exercises (Legacy Textbooks)

Ex 5.5 (Arnold & Arnold, 2000)

- Q1-14

Ex 2B (Patel, 2004)

Ex 4.7 (Lee, 2006)

Section 7

Trigonometric integrals

7.1 Power of $\sin x$, $\cos x$

Learning Goal(s)

Knowledge
Integrating trigonometric integrals with the toolkit previously learned

Skills
Identifying which type of trigonometric integral

Understanding
There may be different primitives but they may just differ by a constant of integration

By the end of this section am I able to:

27.1 Find and evaluate indefinite and definite integrals using the method of integration by substitution, using a given substitution

- (x2) Substitution may not be given!

Theorem 2

To evaluate $\int \cos^m x \sin^n x \, dx$:

- If m, n are both even, use double angle formulae.
- Otherwise, use Pythagorean identity $\sin^2 x + \cos^2 x = 1$ and a substitution.

Example 67

Evaluate $\int_0^{\frac{\pi}{2}} 4\cos^2 x \sin^2 x \, dx$. **Answer:** $\frac{\pi}{4}$

Example 68

Determine $\displaystyle\int \cos^3 x \sin^2 x \, dx$.

Answer: $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

Example 69

⚠ Evaluate $\displaystyle\int \sin^5 \theta \cos^4 \theta \, d\theta$.

Answer: $-\frac{1}{5}\cos^5 \theta + \frac{2}{7}\cos^7 \theta - \frac{1}{9}\cos^9 \theta + C$

Important note

These types are generally not in the syllabus.

7.2 Power of $\tan x$, $\sec x$

Theorem 3

To evaluate $\displaystyle\int \sec^m x \tan^n x \, dx$:

- If m, n are both even, separate $\sec^2 x$ and make substitution $u = \tan x$ (chain rule in reverse)
- If n odd, factor out $\sec x$, substitute $u = \tan x$.
- If m odd and n even, use integration by parts.
- Otherwise, use Pythagorean identity $1 + \tan^2 x = \sec^2 x$ and manipulate.

Example 70

Evaluate $\displaystyle\int \tan^4 x \, dx$. **Answer:** $x + \frac{1}{3}\tan^3 x - \tan x + C$

Example 71

Show that $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x \tan^2 x \, dx = \frac{8}{15}$.

Example 72

Determine the value of $\displaystyle\int_0^{\frac{\pi}{3}} \sec^3 x \tan x \, dx$. **Answer:** $\frac{7}{3}$

Example 73

Evaluate $\displaystyle\int \sec^3 x \, dx$. **Answer:** $\frac{1}{2}(\sec x \tan x + \ln(\sec x + \tan x)) + C$

7.3 t-formulae

- If all else fails, make substitution $t = \tan \frac{x}{2}$

Example 74

Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{4}{3 + 5\cos x}\, dx = \ln 3$.

Example 75

Evaluate $\displaystyle\int \frac{\cos x}{3 + 2\cos x}\,dx$.

Answer: $\frac{x}{2} - \frac{3}{\sqrt{5}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{5}}\right) + C$

Example 76

[2011 Independent Ext 2 Trial]

(i) Use the substitution $t = \tan \frac{x}{2}$, show that **2**

$$\int_0^{\frac{\pi}{2}} \frac{1}{5 + 5\sin x - 3\cos x}\, dx = \int_0^1 \frac{1}{4t^2 + 5t + 1}\, dt$$

(ii) Hence evaluate in simplest exact form $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{5 + 5\sin x - 3\cos x}\, dx$. **2**

Answer: $\frac{1}{3}\ln\frac{5}{2}$

Further exercises

Ex 4G (Sadler & Ward, 2019)

- Q1-16

Further exercises (Legacy Textbooks)

Ex 5.4 (Arnold & Arnold, 2000)
Ex 2.4 (Patel, 1990)
Ex 2C (Patel, 2004)
Ex 4.5 (Lee, 2006)

Section 8

Reduction formulae

Learning Goal(s)

Knowledge
What is a reduction formula/recurrence relation

Skills
Developing the reduction formula/recurrence relation

Understanding
Why reduction formulae/recurrent relationships are required

By the end of this section am I able to:
27.10 Derive and use recurrence relationships

8.1 Rationale

- Some integrals with higher powers are notoriously difficult to handle.
- Of these, some can be written as a simpler integral with changes in index, constants by observing an integral with lesser power inside the current integral.
- Use reduction formulae in reverse to obtain answer to original question.

8.2 Using trigonometric identities

Example 77

Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x \, dx$.

(a) Show that $I_n = \dfrac{1}{n-1} - I_{n-2}$ for $n \geq 2$.

(b) Evaluate I_1 and hence, I_5.

Solution

Steps

(a)

- Factor out $\tan^2 x$ from original integrand:

- Expand, split integral into two, one of which is an integral with two less in the power.

- Evaluate other definite integral

(b)

- Evaluate I_1:

- Evaluate I_3, and hence I_5:

8.3 Using integration by parts

Important note

Most reduction formulae questions require integration by parts.

Example 78

(a) Let $I_n = \int_1^e (\log_e x)^n \, dx$, and show that $I_n = e - nI_{n-1}$ for $n \geq 1$.

(b) Find I_0 and hence show that $I_3 = 6 - 2e$.

Example 79

[2013 Ext 2 HSC Q13] Let $I_n = \int_0^1 \left(1 - x^2\right)^{\frac{n}{2}}\, dx$, where $n \in \mathbb{N}$.

(a) Show that $I_n = \dfrac{n}{n+1} I_{n-2}$, for every $n \in \mathbb{Z}$, $n \geq 2$. **3**

(b) Evaluate I_5. **2**

Answer: $\frac{5\pi}{32}$

Example 80

[2020 Ext 2 HSC Sample Q16] Let $I_n = \displaystyle\int_0^1 x^n\sqrt{1-x^2}\,dx$, for $n = 0, 1, 2, \ldots$.

i. Find the value of I_1. **1**

ii. Using integration by parts, or otherwise, show that for $n > 2$ **3**

$$I_n = \left(\frac{n-1}{n+2}\right) I_{n-2}$$

iii. Find the value of I_5. **1**

Answer: $\frac{8}{105}$

Example 81

[2016 CSSA Ext 2 Trial] Consider the integral $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$, where n is a positive integer.

i. Find I_0. **1**

ii. Show that $I_{n-1} + I_n = \displaystyle\int_0^1 x^{n-1}\sqrt{1+x}\,dx$. **1**

iii. Use integration by parts to show that $I_n = \dfrac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$. **2**

Further exercises

Ex 4H (Sadler & Ward, 2019)

- Q1-14

Further exercises (Legacy Textbooks)

Ex 5.5 (Arnold & Arnold, 2000)

- Q15-20

Ex 4.8 (Lee, 2006)

Section 9

Further properties

9.1 'Dummy' variable

Theorem 4

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(\theta)\,d\theta$$

(Letter inside does not matter, provided f is the same function)

Example 82

What is the value of $\int_a^b x^2\,dx = \int_a^b t^2\,dt = \int_a^b \theta^2\,d\theta$?

Example 83

(a) Evaluate $\int_1^x t^2\,dt$.

(b) Write a question involving the same integrand, when evaluated, produces the same result.

9.2 Reflection about $x = \frac{a}{2}$

Learning Goal(s)

Knowledge
The reflection property about $x = \frac{a}{2}$

Skills
Identifying the reflection point to obtain a similar integral

Understanding
When to use the reflection property

By the end of this section am I able to:

27.1 Find and evaluate indefinite and definite integrals using the method of integration by substitution, using a given substitution

- (x2) Substitution may not be given!

Theorem 5

For a function f continuous between $x = 0$ and $x = a$,

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Note: $f(a-x) = f(-x+a) = f(-(x-a))$: use transformations to reflect $f(x)$ about y axis, and then shift right by a units, giving symmetry about $x = \frac{a}{2}$.

Proof Given $\displaystyle\int_0^a f(x)\,dx$, make the substitution $u = a - x$:

Example 84

Let $I = \displaystyle\int_0^{\pi} x \sin x\,dx$. Evaluate this integral by determining a suitable reflection.

Answer: π

Example 85

Use the property $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ to determine the value of

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$

Answer: $\frac{\pi}{4}$

Example 86

Use the property $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ to determine the value of

$$\int_0^{2} x^2\sqrt{2-x}\,dx$$

Answer: $\frac{128\sqrt{2}}{105}$

Example 87

[2019 Independent Ext 2 Trial Q15]

i. Prove $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **1**

ii. Hence evaluate $\displaystyle\int_0^1 \left(e^{\frac{1}{2}-x} + e^{x-\frac{1}{2}}\right)\sin^2\left(\frac{\pi}{2}x\right)\,dx$. **3**

Answer: $\frac{e-1}{\sqrt{e}}$

Further exercises (Legacy Textbooks)

Ex 4.6 (Lee, 2006)

9.2.1 Further manipulation

- Manipulate by adding constants and subtracting them immediately.

Example 88

Let $I_n = \displaystyle\int_0^1 x^2 \left(1 - x^2\right)^n \, dx$.

(a) Use the identity $x^2 \equiv 1 - \left(1 - x^2\right)$ to show that $I_n = \dfrac{2n}{2n+3} I_{n-1}$ for $n \geq 1$.

(b) Evaluate I_0 and hence find I_3.

Answer: $\frac{16}{315}$

9.3 Bounding

Theorem 6

Suppose between $a \leq x \leq b$, that $f(x) \leq g(x) \leq h(x)$. Then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx \leq \int_a^b h(x)\,dx$$

Example 89

(a) Prove that $\dfrac{1}{x+1} \leq \dfrac{1}{x+\cos^2 x} \leq \dfrac{1}{x}$ for $x > 0$.

(b) Hence show that $\log_e \frac{3}{2} \leq \displaystyle\int_1^2 \frac{1}{x+\cos^2 x}\,dx \leq \log_e 2$.

Example 90

[2020 Ext 2 HSC Sample Q16/2014 Ext 2 HSC Q16] Suppose n is a positive integer.

i. Show that $-x^{2n} \leq \dfrac{1}{1+x^2} - \sum_{k=0}^{n-1} (-1)^k x^{2k} \leq x^{2n}$. **3**

ii. Use integration to deduce that **3**

$$-\frac{1}{2n+1} \leq \frac{\pi}{4} - \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \leq \frac{1}{2n+1}$$

iii. Hence deduce the value of $\sum_{k=0}^{\infty} \dfrac{(-1)^k}{2k+1}$. **1**

Answer: $\frac{\pi}{4}$

Important note

⚠ Beware of multidisciplinary question from elsewhere in the course. **MEX-P1 The Nature of Proof** hides in this question, and many others.

Further exercises

Ex 4I (Sadler & Ward, 2019)

- Q1-14

Further exercises (Legacy Textbooks)

Ex 5.6 (Arnold & Arnold, 2000)
Ex 2I (Patel, 2004)

Part III

Appendices

Section A

Past HSC Questions

A.1 1995 HSC

Question 1

(a) Find $\displaystyle\int \frac{dx}{x\,(\ln x)^2}$. **2**

(b) Find $\displaystyle\int xe^x\,dx$. **2**

(c) Show that $\displaystyle\int_1^4 \frac{6t+23}{(2t-1)(t+6)}\,dt = \ln 70$. **4**

(d) Find $\displaystyle\frac{d}{dx}\left(x\sin^{-1}x\right)$, and hence find $\displaystyle\int \sin^{-1}x\,dx$. **3**

(e) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, calculate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{5+3\sin x+4\cos x}$. **4**

Question 4

(c) i. Show that, if $0 < x < \dfrac{\pi}{2}$, then **2**

$$\frac{\sin(2m+1)x}{\sin x} - \frac{\sin(2m-1)x}{\sin x} = 2\cos(2mx)$$

ii. Show that, for any positive integer m, **1**

$$\int_0^{\frac{\pi}{2}} \cos(2mx)\,dx = 0$$

iii. Deduce that, if m is any positive integer, **1**

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2m+1)x}{\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin(2m-1)x}{\sin x}\,dx$$

iv. Show that, if $m = 1$, then **1**

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2m-1)x}{\sin x}\,dx = \frac{\pi}{2}$$

v. Hence show that **2**

$$\int_0^{\frac{\pi}{2}} \frac{\sin 5x}{\sin x}\, dx = \frac{\pi}{2}$$

Question 7

(a) Let $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n \, dx$, where n is an integer, $n \geq 0$.

i. Using integration by parts, show that, for $n \geq 2$, **3**

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

ii. Deduce that **3**

$$I_{2n} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \cdots \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

iii. Explain why $I_k > I_{k+1}$. **1**

iv. Hence, using the fact that $I_{2n-1} > I_{2n}$ and $I_{2n} > I_{2n+1}$, show that **1**

$$\frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2 \times 4^2 \cdots (2n)^2}{1 \times 3^2 \times 5^2 \cdots (2n-1)^2(2n+1)} < \frac{\pi}{2}$$

A.2 1996 HSC

Question 1

(a) Evaluate $\int_1^3 \frac{4}{(2+x)^2}\, dx$. **2**

(b) Find $\int \sec^2\theta \tan\theta \, d\theta$. **2**

(c) Find $\int \frac{5t^2+3}{t\,(t^2+1)}\, dt$. **3**

(d) Using integration by parts, or otherwise, find $\int x \tan^{-1} x \, dx$. **3**

(e) Using the substitution $x = 2\sin\theta$, or otherwise, calculate $\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\, dx$. **5**

Question 3

(c) i. Show that $\int_0^{\frac{\pi}{2}} (\sin x)^{2k} \cos x \, dx = \frac{1}{2k+1}$, where k is a positive integer. **1**

ii. By writing $(\cos x)^{2n} = \left(1 - \sin^2 x\right)^n$, show that **4**

$$\int_0^{\frac{\pi}{2}} (\cos x)^{2n+1} \, dx = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\binom{n}{k}$$

iii. Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x \, dx$ **1**

Question 7

(a) i. Let $f(x) = \ln x - ax + b$, for $x > 0$, where a and b are real numbers and $a > 0$. Show that $y = f(x)$ has a single turning point which is a maximum. **2**

ii. The graphs of $y = \ln x$ and $y = ax - b$ intersect at points A and B. Using the result of part (i), or otherwise, show that the chord AB lies below the curve $y = \ln x$. **1**

iii. Using integration by parts, or otherwise, show that **1**

$$\int_1^k \ln x\, dx = k \ln k - k + 1$$

iv. Use the trapezoidal rule on the intervals with integer endpoints $1, 2, 3, \ldots, k$ to show that **2**

$$\int_1^k \ln x\, dx \approx \frac{1}{2} \ln k + \ln\left[(k-1)!\right]$$

v. Hence deduce that **3**

$$k! < e\sqrt{k}\left(\frac{k}{e}\right)^k$$

A.3 1997 HSC

Question 1

(a) Evaluate $\displaystyle\int_0^5 \frac{2}{\sqrt{x+4}}\, dx$ **2**

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^4\theta}\, d\theta$. **3**

(c) Find $\displaystyle\int \frac{1}{x^2+2x+3}\, dx$. **2**

(d) Find $\displaystyle\int \frac{4t-6}{(t+1)(2t^2+3)}\, dt$. **4**

(e) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} x \sec^2 x\, dx$. **4**

Question 6

(a) The series $1 - x^2 + x^4 - \cdots + x^{4n}$ has $2n+1$ terms.

i. Explain why **2**

$$1 - x^2 + x^4 - \cdots + x^{4n} = \frac{1 + x^{4n+2}}{1+x^2}$$

ii. Hence show that **2**

$$\frac{1}{1+x^2} \leq 1 - x^2 + x^4 - \cdots + x^{4n} \leq \frac{1}{1+x^2} + x^{4n+2}$$

iii. Hence show that, if $0 \leq y \leq 1$, then **2**

$$\tan^{-1} y \leq y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots + \frac{y^{4n+1}}{4n+1} \leq \tan^{-1} y + \frac{1}{4n+3}$$

iv. Deduce that **1**

$$0 < \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{1001}\right) - \frac{\pi}{4} < 10^{-3}$$

A.4 1998 HSC

Question 1

(a) Evaluate $\displaystyle\int_0^3 \frac{6}{9+x^2}\,dx$. **2**

(b) Find $\displaystyle\int x^2 \ln x\,dx$. **2**

(c) Find $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,dx$. **3**

(d) Using the substitution $u^2 = 4 - x^2$ or otherwise, evaluate $\displaystyle\int_0^2 x^3\sqrt{4-x^2}\,dx$. **4**

(e) i. Find the remainder when x^2+6 is divided by x^2+x-6. **2**

ii. Hence find $\displaystyle\int \frac{x^2+6}{x^2+x-6}\,dx$. **2**

Question 3

(b) Let $I_n = \displaystyle\int_1^e (\ln x)^n\,dx$.

i. Show that $I_n = e - nI_{n-1}$ for $n = 1, 2, 3, \ldots$. **2**

ii. Hence evaluate I_4. **2**

Question 7

(b) i. Differentiate $\sin^{-1}(u) - \sqrt{1-u^2}$. **2**

ii. Hence show that $\displaystyle\int_0^\alpha \left(\frac{1+u}{1-u}\right)^{\frac{1}{2}} du = \sin^{-1}\alpha + 1 - \sqrt{1-\alpha^2}$ for $0 < \alpha < 1$. **1**

A.5 1999 HSC

Question 1

(a) Evaluate $\displaystyle\int_0^1 xe^{-x^2}\,dx$. **2**

(b) Using the substitution $u = e^x$ or otherwise, find $\displaystyle\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$. **2**

(c) Find $\displaystyle\int \frac{4x^3-2x^2+1}{2x-1}\,dx$. **3**

(d) i. Find constants a, b and c such that $\dfrac{x^2+2x}{(x^2+4)(x-2)} = \dfrac{ax+b}{x^2+4} + \dfrac{c}{x-2}$. **2**

ii. Hence find $\displaystyle\int \frac{x^2+2x}{(x^2+4)(x-2)}\,dx$. **2**

(e) Use integration by parts to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x^2 \sin x\,dx$. **4**

Question 7

(a) Graph $y = \ln x$ and draw the tangent to the graph at $x = 1$. **1**

(b) By considering the appropriate area under the tangent, deduce that **2**

$$\int_1^{\frac{3}{2}} \ln x\,dx \le \frac{1}{8}$$

(c) By considering the graph of $y = \ln x$, explain why **3**

$$\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \ln x\,dx \le \ln k$$

for $k = 2, 3, 4, \ldots$.

(d) Deduce that **2**

$$\int_1^n \ln x\,dx \le \frac{1}{8} + \ln 2 + \ln 3 + \cdots + \ln(n-1) + \frac{1}{2}\ln n$$

for $n = 2, 3, 4, \cdots$.

(e) Assuming that $\displaystyle\int_1^n \ln x\,dx = n\ln n - n + 1$, deduce that **2**

$$n! \ge e^{\frac{7}{8}} n^n \sqrt{n} e^{-n}$$

for $n = 2, 3, 4, \cdots$.

A.6 2000 HSC

Question 1

(a) Find $\displaystyle\int \frac{\cos x}{\sin^4 x}\,dx$. **2**

(b) Use completion of squares to find $\displaystyle\int \frac{4}{x^2+6x+10}\,dx$. **2**

(c) i. Find the real number a, b and c such that $\dfrac{9}{x^2(3-x)} \equiv \dfrac{ax+b}{x^2} + \dfrac{c}{3-x}$. **2**

ii. Find $\displaystyle\int \frac{9}{x^2(3-x)}\,dx$. **2**

(d) Find $\int \sqrt{x} \ln x \, dx$. **3**

(e) Use the substitution $t = \tan \frac{\theta}{2}$ to find $\int \frac{d\theta}{1 + \sin\theta + \cos\theta}$. **4**

Question 6

(b) Evaluate $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}}$. **2**

(c) Explain carefully why, for $n \geq 2$, **4**

$$\frac{1}{2} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^n}} \leq \frac{\pi}{6}$$

A.7 2001 HSC

Question 1

(a) Find $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx$. **2**

(b) By completing the square, find $\int \frac{dx}{\sqrt{x^2 - 4x + 1}}$. **2**

(c) Use integration by parts to evaluate $\int_e^4 \frac{\ln x}{x^2} \, dx$. **3**

(d) Use the substitution $u = \sqrt{x - 1}$ to evaluate **4**

$$\int_2^3 \frac{1 + x}{\sqrt{x - 1}} \, dx$$

(e) i. Find real numbers a and b such that **2**

$$\frac{5x^2 - 3x + 1}{(x^2 + 1)(x - 2)} \equiv \frac{ax + 1}{x^2 + 1} + \frac{b}{x - 2}$$

ii. Find $\int \frac{5x^2 - 3x + 1}{(x^2 + 1)(x - 2)} \, dx$. **2**

Question 8

(b) i. Explain why, for $\alpha > 0$, **2**

$$\int_0^1 x^\alpha e^x \, dx < \frac{3}{\alpha + 1}$$

(You may assume $e < 3$)

ii. Show, by induction, that for $n = 0, 1, 2, \ldots$, there exist integers a_n and b_n such that **2**

$$\int_0^1 x^n e^x \, dx = a_n + b_n e$$

iii. Suppose that r is a positive rational, so that $r = \dfrac{p}{q}$ where p and q are positive integers. Show that, for all integers a and b, either **2**

$$|a + br| = 0 \quad \text{or} \quad |a + br| \geq \frac{1}{q}$$

iv. Prove that e is irrational. **2**

A.8 2002 HSC

Question 1

(a) By using the substitution $u = \sec x$, or otherwise, find **2**

$$\int \sec^3 x \tan x \, dx$$

(b) By completing the square, find $\displaystyle\int \frac{dx}{x^2 + 2x + 2}$. **2**

(c) Find $\displaystyle\int \frac{x\, dx}{(x+3)(x-1)}$. **3**

(d) By using two applications of integration by parts, evaluate **4**

$$\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

(e) Use the substitution $t = \tan \frac{\theta}{2}$ to find **4**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2 + \cos\theta}$$

Question 6

(b) i. For $n = 0, 1, 2 \ldots$ let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n \theta \, d\theta$$

ii. Show that $I_1 = \frac{1}{2} \ln 2$. **1**

iii. Show that, for $n \geq 2$, **3**

$$I_n + I_{n-2} = \frac{1}{n-1}$$

iv. For $n \geq 2$, explain why $I_n < I_{n-2}$, and deduce that **3**

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$$

v. By using the recurrence relation of part (ii), find I_5 and deduce that **2**

$$\frac{2}{3} < \ln 2 < \frac{3}{4}$$

A.9 2003 HSC

Question 1

(a) Evaluate $\displaystyle\int \frac{e^x}{(1+e^x)^2}\,dx$. **2**

(b) Use integration by parts to find **3**

$$\int x^3 \log_e x\,dx$$

(c) By completing the square and using the table of standard integrals, find **2**

$$\int \frac{dx}{\sqrt{x^2-2x+5}}$$

(d) i. Find real numbers a and b such that **2**

$$\frac{5x^2-3x+13}{(x-1)(x^2+4)} \equiv \frac{a}{x-1}+\frac{bx-1}{x^2+4}$$

ii. Find $\displaystyle\int \frac{5x^2-3x+13}{(x-1)(x^2+4)}\,dx$. **2**

(e) Use the substitution $x = 3\sin\theta$ to evaluate **4**

$$\int_0^{\frac{3}{\sqrt{2}}} \frac{dx}{(9-x^2)^{\frac{3}{2}}}$$

Question 8

(b) Suppose that π could be written in the form $\frac{p}{q}$, where p and q are positive integers. Define the family of integratls I_n for $n = 0, 1, 2\ldots$ by

$$I_n = \frac{q^{2n}}{n!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4}-x^2\right)^n \cos x\,dx$$

You are given that $I_0 = 2$ and $I_1 = 4q^2$ (Do NOT prove this).

i. Use integration by parts to twice to show that for $n \geq 2$, **3**

$$I_n = \frac{2q^{2n}}{(n-1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4}-x^2\right)^{n-1} \cos x\,dx - \frac{4q^{2n}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\left(\frac{\pi^2}{4}-x^2\right)^{n-2} \cos x\,dx$$

ii. By writing x^2 as $\frac{\pi^2}{4} - \left(\frac{\pi^2}{4}-x^2\right)$ where appropriate, deduce that **1**

$$I_n = (4n-2)q^2 I_{n-1} - p^2q^2 I_{n-2}$$

for $n \geq 2$.

iii. Explain briefly why I_n is an integer for $n = 0, 1, 2, \ldots$. **1**

iv. Prove that **2**

$$0 < I_n < \frac{p}{q}\left(\frac{p}{2}\right)^{2n}\frac{1}{n!}$$

for $n = 0, 1, 2 \ldots$

v. Given that $\frac{p}{q}\left(\frac{p}{2}\right)^{2n}\frac{1}{n!} < 1$, if n is sufficiently large, deduce that π is irrational. **1**

A.10 2004 HSC

Question 1

(a) Use integration by parts to find $\int xe^{3x}\,dx$. **2**

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,dx$. **3**

(c) By completing the square, find $\int \frac{dx}{\sqrt{5+4x-x^2}}$. **2**

(d) i. Find real numbers a and b such that **2**

$$\frac{x^2-7x+4}{(x+1)(x-1)^2} \equiv \frac{a}{x+1} + \frac{b}{x-1} - \frac{1}{(x-1)^2}$$

ii. Hence find $\int \frac{x^2-7x+4}{(x+1)(x-1)^2}\,dx$. **2**

(e) Use the substitution $x = 2\sin\theta$ to find $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}$. **4**

Question 6

(a) i. Show that **2**

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}$$

ii. By making the substitution $x = \pi - u$, find **3**

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

Question 8

(b) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$ and let $J_n = (-1)^n I_{2n}$ for $n = 0, 1, 2 \ldots$.

i. Show that $I_n + I_{n+2} = \frac{1}{n+1}$. **2**

ii. Deduce that $J_n - J_{n-1} = \frac{(-1)^n}{2n-1}$ for $n \geq 1$. **1**

iii. Show that $J_m = \frac{\pi}{4} + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$. **2**

iv. Use the substitution $u = \tan x$ to show that $I_n = \displaystyle\int_0^1 \frac{u^n}{1+u^2}\,du$. **2**

v. Deduce that $0 \le I_n \le \dfrac{1}{n+1}$ and conclude that $J_n \to 0$ as $n \to \infty$. **2**

A.11 2005 HSC

Question 1

(a) Find $\displaystyle\int \frac{\cos\theta}{\sin^5\theta}\,d\theta$. **2**

(b) Find real numbers a and b such that $\dfrac{5x}{x^2-x-6} \equiv \frac{a}{x-3} + \frac{b}{x+2}$. **2**

(c) Hence find $\displaystyle\int \frac{5x}{x^2-x-6}\,dx$. **1**

(d) Use integration by parts to evaluate $\displaystyle\int_1^e x^7 \log_e x\,dx$. **3**

(e) Using the table of standard integrals, or otherwise, find $\displaystyle\int \frac{dx}{\sqrt{4x^2-1}}$. **2**

(f) Let $t = \tan\frac{\theta}{2}$.

i. Show that $\dfrac{dt}{d\theta} = \dfrac{1}{2}\left(1+t^2\right)$. **1**

ii. Show that $\sin\theta = \dfrac{2t}{1+t^2}$. **2**

iii. Use the substitution $t = \tan\frac{\theta}{2}$ to find $\displaystyle\int \operatorname{cosec}\theta\,d\theta$. **2**

Question 5

(c) Let $a > 0$ and let $f(x)$ be an increasing function such that $f(0) = 0$ and $f(a) = b$.

i. Explain why $\displaystyle\int_0^a f(x)\,dx = ab - \int_0^b f^{-1}(x)\,dx$. **1**

ii. Hence or otherwise, find the value of $\displaystyle\int_0^2 \sin^{-1}\left(\frac{x}{4}\right)\,dx$. **3**

Question 6

(a) For each integer $n \ge 0$, let $I_n(x) = \displaystyle\int_0^x t^n e^{-t}\,dt$.

i. Prove by induction that **4**

$$I_n(x) = n!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)\right]$$

ii. Show that **1**

$$0 \le \int_0^1 t^n e^{-t}\,dt \le \frac{1}{n+1}$$

iii. Hence show that **1**

$$0 \leq 1 - e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots \frac{1}{n!}\right) \leq \frac{1}{(n+1)!}$$

iv. Hence find the limiting value of $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ as $n \to \infty$. **1**

A.12 2006 HSC

Question 1

(a) Find $\int \frac{x}{\sqrt{9-4x^2}}\,dx$. **2**

(b) By completing the square, find $\int \frac{dx}{x^2 - 6x + 13}$. **3**

(c) i. Given that $\frac{16x-43}{(x-3)^2(x+2)}$ can be written as **2**

$$\frac{16x-43}{(x-3)^2(x+2)} = \frac{a}{(x-3)^2} + \frac{b}{x-3} + \frac{c}{x+2}$$

where a, b, $c \in \mathbb{R}$, find a, b and c.

ii. Evaluate $\int_0^2 te^{-t}\,dt$. **3**

(d) Use the substitution $t = \tan\frac{\theta}{2}$ to show that **3**

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{\sin\theta} = \frac{1}{2}\log 3$$

Question 7

(b) i. Let $I_n = \int_0^x \sec^n t\,dt$ where $0 \leq x \leq \frac{\pi}{2}$. Show that **3**

$$I_n = \frac{\sec^{n-1} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

ii. Hence find the exact value of **2**

$$\int_0^{\frac{\pi}{3}} \sec^4 t\,dt$$

Question 8

(a) Suppose $0 \leq t \leq \frac{1}{\sqrt{2}}$

i. Show that $0 \leq \frac{2t^2}{1-t^2} \leq 4t^2$. **2**

ii. Hence show that $0 \leq \frac{1}{1+t} + \frac{1}{1-t} - 2 \leq 4t^2$. **1**

iii. By integrating the expressions in the inequality in part (ii) with respect to t from $t = 0$ to $t = x$, (where $0 \leq x \leq \frac{1}{\sqrt{2}}$), show that **2**

$$0 \leq \log_e \left(\frac{1+x}{1-x} \right) - 2x \leq \frac{4x^3}{3}$$

iv. Hence show that for $0 \leq x \leq \frac{1}{\sqrt{2}}$, **1**

$$1 \leq \left(\frac{1+x}{1-x} \right) e^{-2x} \leq e^{\frac{4x^3}{3}}$$

A.13 2007 HSC

Question 1

(a) Find $\displaystyle\int \frac{1}{\sqrt{9 - 4x^2}} \, dx$. **2**

(b) Find $\displaystyle\int \tan^2 x \sec^2 x \, dx$. **2**

(c) Evaluate $\displaystyle\int_0^{\pi} x \cos x \, dx$. **3**

(d) Evaluate $\displaystyle\int_0^{\frac{3}{4}} \frac{x}{\sqrt{1-x}} \, dx$. **4**

(e) It can be shown that **4**

$$\frac{2}{x^3 + x^2 + x + 1} = \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}$$

(Do NOT prove this).

Use this result to evaluate $\displaystyle\int_{\frac{1}{2}}^{2} \frac{2}{x^3 + x^2 + x + 1} \, dx$.

Question 5

(c) i. Write $(x-1)(5-x)$ in the form $b^2 - (x-a)^2$, where a and b are real numbers. **1**

ii. Using the values of a and b found in part (i) and making the substitution $x - a = b \sin \theta$, or otherwise, evaluate **2**

$$\int_1^5 \sqrt{(x-1)(5-x)} \, dx$$

Question 8

(a) i. Using a suitable substitution, show that **1**

$$\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$$

ii. A function has property $f(x) + f(a-x) = f(a)$. Using part (i) or otherwise, show that **2**

$$\int_0^a f(x) \, dx = \frac{a}{2} f(a)$$

A.14 2008 HSC

Question 1

(a) Find $\displaystyle\int \frac{x^2}{(5+x^3)^2}\,dx$ **2**

(b) Find $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$. **2**

(c) Evaluate $\displaystyle\int_0^1 \tan^{-1} x\,dx$. **3**

(d) Evaluate $\displaystyle\int_1^2 \frac{dx}{x\sqrt{2x-1}}$. **4**

(e) It can be shown that **4**

$$\frac{8(1-x)}{(2-x^2)(2-2x+x^2)} = \frac{4-2x}{2-2x+x^2} - \frac{2x}{2-x^2}$$

(Do NOT prove this).

Use this result to evaluate $\displaystyle\int_0^1 \frac{8(1-x)}{(2-x^2)(2-2x+x^2)}\,dx$.

Question 3

(c) For $n \geq 0$, let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^{2n}\theta\,d\theta$$

i. Show that for $n \geq 1$, **2**

$$I_n = \frac{1}{2n-1} - I_{n-1}$$

ii. Hence or otherwise, calculate I_3. **2**

A.15 2009 HSC

Question 1

(a) Find $\displaystyle\int \frac{\ln x}{x}\,dx$. **2**

(b) Find $\displaystyle\int xe^{2x}\,dx$. **2**

(c) Find $\displaystyle\int \frac{x^2}{1+4x^2}\,dx$. **3**

(d) Evaluate $\displaystyle\int_2^5 \frac{x-6}{x^2+3x-4}\,dx$. **4**

(e) Evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx$. **4**

Question 5

(b) For each integer $n \geq 0$, let

$$I_n = \int_0^1 x^{2n+1} e^{x^2}\, dx$$

i. Show that for $n \geq 1$, **2**

$$I_n = \frac{e}{2} - nI_{n-1}$$

ii. Hence or otherwise, calculate I_2. **2**

Question 7

(b) Let $z = \cos\theta + i\sin\theta$.

i. Show that $z^n + z^{-n} = 2\cos n\theta$, where n is a positive integer. **2**

ii. Let m be a positive integer. Show that **3**

$$(2\cos\theta)^{2m} = 2\left[\cos 2m\theta + \binom{2m}{1}\cos(2m-2)\theta + \binom{2m}{2}\cos(2m-4)\theta + \cdots + \binom{2m}{m-1}\cos 2\theta\right] + \binom{2m}{m}$$

iii. Hence, or otherwise, prove that **2**

$$\int_0^{2m} \cos^{2m}\theta d\theta = \frac{\pi}{2^{2m+1}}\binom{2m}{m}$$

where m is a positive integer.

A.16 2010 HSC

Question 1

(a) Find $\displaystyle\int \frac{x}{\sqrt{1+3x^2}}\, dx$. **2**

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan x\, dx$. **3**

(c) Find $\displaystyle\int \frac{1}{x\left(x^2+1\right)}\, dx$. **3**

(d) Using the substitution $t = \tan\frac{x}{2}$ or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$. **4**

(e) Find $\displaystyle\int \frac{dx}{1+\sqrt{x}}$. **3**

Question 8

Let

$$A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x\, dx \quad \text{and} \quad B_n = \int_0^{\frac{\pi}{2}} x^2\cos^{2n} x\, dx$$

where $n \in \mathbb{Z}$, $n \geq 0$. (Note that $A_n > 0$, $B_n > 0$.)

(a) Show that $nA_n = \dfrac{2n-1}{2}A_{n-1}$ for $n \geq 1$. **2**

(b) Using integration by parts on A_n, or otherwise, show that **1**

$$A_n = 2n\int_0^{\frac{\pi}{2}} x\sin x\cos^{2n-1} x\,dx$$

for $n \geq 1$.

(c) Use integration by parts on the integral in part (b) to show that **3**

$$\frac{A_n}{n^2} = \frac{(2n-1)}{n}B_{n-1} - 2B_n$$

for $n \geq 1$.

(d) Use parts (a) and (c) to show that **1**

$$\frac{1}{n^2} = 2\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right)$$

for $n \geq 1$.

(e) Show that $\displaystyle\sum_{k=1}^{n}\frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n}$. **2**

(f) Use the fact that $\sin x \geq \frac{2}{\pi}x$ for $0 \leq x \leq \frac{\pi}{2}$ to show that **1**

$$B_n \leq \int_0^{\frac{\pi}{2}} x^2\left(1-\frac{4x^2}{\pi^2}\right)^n\,dx$$

(g) Show that $\displaystyle\int_0^{\frac{\pi}{2}} x^2\left(1-\frac{4x^2}{\pi^2}\right)^n\,dx = \frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}\,dx$. **1**

(h) From parts (f) and (g) it follows that **2**

$$B_n \leq \frac{\pi^2}{8(n+1)}\int_0^{\frac{\pi}{2}}\left(1-\frac{4x^2}{\pi^2}\right)^{n+1}\,dx$$

Use the substitution $x = \frac{\pi}{2}\sin t$ in this inequality to show that

$$B_n \leq \frac{\pi^3}{16(n+1)}\int_0^{\frac{\pi}{2}}\cos^{2n+3} t\,dt \leq \frac{\pi^3}{16(n+1)}A_n$$

(i) Use part (e) to deduce that **1**

$$\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \leq \sum_{k=1}^{n}\frac{1}{k^2} < \frac{\pi^2}{6}$$

(j) What is $\displaystyle\lim_{x\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}$? **1**

A.17 2011 HSC

Question 1

(a) Find $\int x \ln x \, dx$. **2**

(b) Evaluate $\int_0^3 x\sqrt{x+1}\, dx$. **3**

(c) i. Find a, b and $c \in \mathbb{R}$ such that **2**

$$\frac{1}{x^2(x-1)} \equiv \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}$$

ii. Hence, find $\int \frac{1}{x^2(x-1)}\, dx$. **2**

(d) Find $\int \cos^3 \theta \, d\theta$. **3**

(e) Evaluate $\int_{-1}^{1} \frac{1}{5-2t+t^2}\, dt$. **3**

Question 7

(b) Let $I = \int_1^3 \frac{\cos^2\left(\frac{\pi}{8}x\right)}{x(4-x)}\, dx$.

i. Use the substitution $u = 4 - x$ to show that **2**

$$I = \int_1^3 \frac{\sin^2\left(\frac{\pi}{8}u\right)}{u(4-u)}\, du$$

ii. Hence, find the value of I. **3**

Question 8

(a) For every integer $m \geq 0$, let **3**

$$I_m = \int_0^1 x^m \left(x^2 - 1\right)^5 \, dx$$

Prove that for $m \geq 2$,

$$I_m = \frac{m-1}{m+11} I_{m-2}$$

A.18 2012 HSC

10. Without evaluating the integrals, which of the following is greater than zero? **1**

(A) $\int_{-1}^{1} \frac{x}{2+\cos x}\, dx$

(B) $\int_{-\pi}^{\pi} x^3 \sin x \, dx$

(C) $\int_{-1}^{1} \left(e^{-x^2} - 1\right)\, dx$

(D) $\int_{-2}^{2} \tan^{-1}\left(x^3\right)\, dx$

Question 11

(c) By completing the square, find $\displaystyle\int \frac{dx}{x^2+4x+5}$. **2**

(e) Evaluate $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx$. **3**

Question 12

(a) Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, find **3**

$$\int \frac{d\theta}{1-\cos\theta}$$

(c) For every integer $n \geq 0$, let **3**

$$I_n = \int_1^{e^2} (\log_e x)^n \; dx$$

Show that for $n \geq 1$,

$$I_n = e^2 2^n - nI_{n-1}$$

Question 14

(a) Find $\displaystyle\int \frac{3x^2+8}{x\left(x^2+4\right)}\,dx$. **3**

A.19 2013 HSC

6. Which expression is equal to $\displaystyle\int \frac{1}{x^2-6x+5}\,dx$? **1**

(A) $\sin^{-1}\left(\dfrac{x-3}{2}\right)+C$

(B) $\cos^{-1}\left(\dfrac{x-3}{2}\right)+C$

(C) $\ln\left(x-3+\sqrt{(x-3)^2+4}\right)+C$

(D) $\ln\left(x-3+\sqrt{(x-3)^2-4}\right)+C$

Question 11

(d) Evaluate $\displaystyle\int_0^1 x^3\sqrt{1-x^2}\,dx$. **3**

Question 12

(a) Using the substitution $t = \tan\frac{x}{2}$,or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\,dx$ **4**

Question 14

(c) i. Given a positive integer n, show that $\sec^{2n}\theta = \displaystyle\sum_{k=0}^{n}\binom{n}{k}\tan^{2k}\theta$ **1**

ii. Hence, by writing $\sec^8\theta$ as $\sec^6\theta\sec^2\theta$, find $\displaystyle\int \sec^8\theta\,d\theta$. **2**

A.20 2014 HSC

7. Which expression is equal to $\displaystyle\int \frac{1}{1-\sin x}\,dx$? **1**

(A) $\tan x - \sec x + c$

(B) $\tan x + \sec x + c$

(C) $\log_e(1-\sin x) + c$

(D) $\dfrac{\log_e(1-\sin x)}{-\cos x} + c$

10. Which integral is necessarily equal to $\displaystyle\int_{-a}^{a} f(x)\,dx$? **1**

(A) $\displaystyle\int_0^a f(x) - f(-x)\,dx$

(B) $\displaystyle\int_0^a f(x) - f(a-x)\,dx$

(C) $\displaystyle\int_0^a f(x-a) + f(-x)\,dx$

(D) $\displaystyle\int_0^a f(x-a) + f(a-x)\,dx$

Question 11

(b) Evaluate $\displaystyle\int_0^{\frac{1}{2}} (3x-1)\cos(\pi x)\ dx$. **3**

Question 12

(b) Let $I_n = \displaystyle\int_0^1 \frac{x^{2n}}{x^2+1}\,dx$, where n is an integer and $n \geq 0$.

i. Show that $I_0 = \dfrac{\pi}{4}$. **1**

ii. Show that $I_n + I_{n-1} = \dfrac{1}{2n-1}$. **2**

iii. Hence, or otherwise, find $\displaystyle\int_0^1 \frac{x^4}{x^2+1}\,dx$. **2**

Question 13

(a) Use the substitution $t = \tan\dfrac{x}{2}$, or otherwise, evaluate **3**

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5}\,dx$$

Question 16

(c) Find $\displaystyle\int \frac{\ln x}{(1+\ln x)^2}\,dx$ **3**

A.21 **2015 HSC**

6. Which expression is equal to $\int x^2 \sin x \, dx$? **1**

(A) $-x^2 \cos x - \int 2x \cos x \, dx$

(B) $-2x \cos x + \int x^2 \cos x \, dx$

(C) $-x^2 \cos x + \int 2x \cos x \, dx$

(D) $-2x \cos x - \int x^2 \cos x \, dx$

Question 11

(f) i. Show that $\cot\theta + \operatorname{cosec}\theta = \cot\left(\dfrac{\theta}{2}\right)$. **2**

ii. Hence, or otherwise, find $\int (\cot\theta + \operatorname{cosec}\theta)\, d\theta$. **1**

Question 14

(a) i. Differentiate $\sin^{n-1}\theta\cos\theta$, expressing the result in terms of $\sin\theta$ only. **2**

ii. Hence, or otherwise, deduce that **2**

$$\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta$$

for $n > 1$

iii. Find $\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$. **1**

A.22 2016 HSC

Question 11

(b) Find $\int xe^{-2x}\,dx$ **3**

Question 12

(a) i. Differentiate $xf(x) - \int xf'(x)\,dx$. **1**

ii. Hence, or otherwise, find $\int \tan^{-1} x\,dx$ **2**

Question 14

(b) Let $I_n = \int_0^1 \frac{x^n}{(x^2+1)^2}\,dx$, for $n = 0, 1, 2, \cdots$

i. Using a suitable substitution, show that $I_0 = \frac{\pi}{8} + \frac{1}{4}$. **3**

ii. Show that $I_0 + I_2 = \frac{\pi}{4}$. **1**

iii. Find I_4. **3**

A.23 2017 HSC

7. It is given that $f(x)$ is a non-zero even function and $g(x)$ is a non-zero odd function. **1**

Which expression equals to $\int_{-a}^{a} (f(x) + g(x))\,dx$?

(A) $2\int_0^a f(x)\,dx$

(B) $2\int_0^a g(x)\,dx$

(C) $\int_{-a}^{a} g(x)\,dx$

(D) $2\int_0^a (f(x) + g(x))\,dx$

10. Suppose $f(x)$ is a differentiable function such that $\frac{f(a)+f(b)}{2} \geq f\left(\frac{a+b}{2}\right)$, for all a and b. **1**

Which statement is always true?

(A) $\int_0^1 f(x)\,dx \geq \frac{f(0)+f(1)}{2}$

(B) $\int_0^1 f(x)\,dx \leq \frac{f(0)+f(1)}{2}$

(C) $f'\left(\frac{1}{2}\right) \geq 0$

(D) $f'\left(\frac{1}{2}\right) \leq 0$

Question 11

(d) Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, evaluate **3**

$$\int_0^{\frac{2\pi}{3}} \frac{1}{1+\cos\theta}\,d\theta$$

(f) Using the substitution $x = \sin^2\theta$, or otherwise, evaluate **3**

$$\int_0^{\frac{1}{2}} \sqrt{\frac{x}{1-x}}\,dx$$

Question 12

(c) Find $\displaystyle\int x\tan^{-1}x\,dx$. **3**

Question 14

(a) It is given that

$$x^4+4=\left(x^2+2x+2\right)\left(x^2-2x+2\right)$$

i. Find A and B so that **1**

$$\frac{16}{x^4+4}=\frac{A+2x}{x^2+2x+2}+\frac{B-2x}{x^2-2x+2}$$

ii. Hence, or otherwise, show that for any real number m, **2**

$$\int_0^m \frac{16}{x^4+4}=\ln\left(\frac{m^2+2m+2}{m^2-2m+2}\right)+2\tan^{-1}(m+1)+2\tan^{-1}(m-1)$$

iii. Find the limiting value as $m\to\infty$ of **1**

$$\int_0^m \frac{16}{x^4+4}\,dx$$

Question 15

(a) Let $I_n=\displaystyle\int_0^1 x^n\sqrt{1-x^2}\,dx$, for $n=0,1,2\cdots$.

i. Find the value of I_1. **1**

ii. Using integration by parts, or otherwise, show that for $n\geq 2$ **3**

$$I_n=\left(\frac{n-1}{n+2}\right)I_{n-2}$$

iii. Find the value of I_5. **1**

A.24 2018 HSC

1. Which expression is equal to $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$? **1**

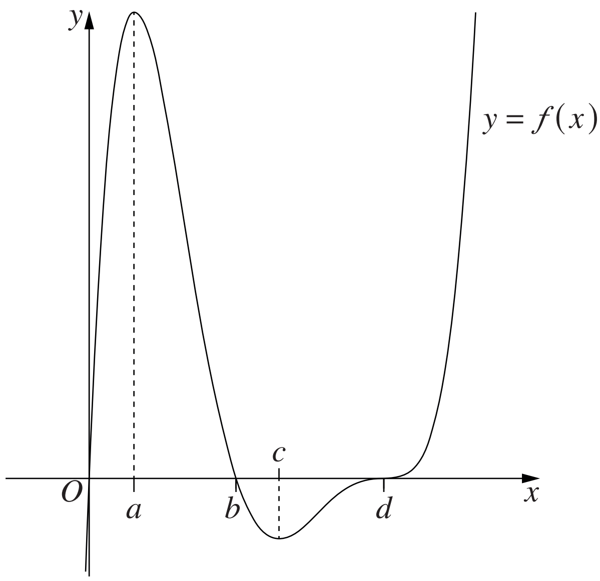
(A) $\frac{1}{2}\sin^{-1}\frac{x}{2}+C$

(B) $\frac{1}{2}\sin^{-1}2x+C$

(C) $\sin^{-1}\frac{x}{2}+C$

(D) $\sin^{-1}2x+C$

8. The diagram shows the graph of the curve $y=f(x)$. **1**



Let $F(x)=\displaystyle\int_0^x f(t)\,dt$.

At what value(s) of x does the concavity of the curve $y=F(x)$ change?

(A) d

(B) a, c

(C) b, d

(D) a, c, d

Question 11

(c) By writing $\dfrac{x^2-x-6}{(x+1)\,(x^2-3)}$ in the form $\dfrac{a}{x+1}+\dfrac{bx+c}{x^2-3}$, **4**

find $\dfrac{x^2-x-6}{(x+1)\,(x^2-3)}\,dx$.

Question 12

(c) Find $\displaystyle\int \frac{x^2+2x}{x^2+2x+5}\,dx$. **3**

Question 14

(a) Using the substitution $t = \tan\frac{\theta}{2}$, evaluate **3**

$$\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos\theta}\,d\theta$$

(c) Let $I_n = \displaystyle\int_{-3}^{0} x^n\sqrt{x+3}\,dx$, for $n = 0, 1, 2\cdots$.

i. Show that, for $n \geq 1$, **3**

$$I_n = \frac{-6n}{3+2n} I_{n-1}$$

ii. Find the value of I_2. **2**

A.25 2019 HSC

2. Which of the following is a primitive of $\dfrac{\sin x}{\cos^3 x}$? **1**

(A) $\dfrac{1}{2}\sec^2 x$ (B) $-\dfrac{1}{2}\sec^2 x$ (C) $\dfrac{1}{4}\sec^4 x$ (D) $-\dfrac{1}{4}\sec^4 x$

3. Which expression is equal to $\displaystyle\int x\cos x\,dx$? **1**

(A) $-x\sin x + \cos x + C$

(B) $-x\sin x - \cos x + C$

(C) $x\sin x + \cos x + C$

(D) $x\sin x - \cos x + C$

7. Which of these integrals has the largest value? **1**

(A) $\displaystyle\int_0^{\frac{\pi}{4}} \tan x\,dx$

(B) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} (1-\tan x)\,dx$

(D) $\displaystyle\int_0^{\frac{\pi}{4}} \left(1-\tan^2 x\right)\,dx$

Question 15

(a) i. Show that **2**

$$\int_{-a}^{a} \frac{f(x)}{f(x)+f(-x)}\,dx = \int_{-a}^{a} \frac{f(-x)}{f(x)+f(-x)}\,dx$$

ii. Hence, or otherwise, evaluate **2**

$$\int_{-1}^{1} \frac{e^x}{e^x+e^{-x}}\,dx$$

(c) i. Show that $\displaystyle\int_0^1 \frac{x}{(x+1)^2}\,dx = \ln 2 - \frac{1}{2}$. **2**

ii. Let $I_n = \displaystyle\int \frac{x^n}{(x+1)^2}\,dx$. **3**

Show that $I_n = \dfrac{1}{2(n-1)} - \dfrac{n}{n-1} I_{n-1}$ for $n \geq 2$.

iii. Evaluate I_3. **2**

A.26 2020 HSC

6. Which expression is equal to $\displaystyle\int \frac{1}{x^2+4x+10}\,dx$? **1**

(A) $\dfrac{1}{\sqrt{6}}\tan^{-1}\left(\dfrac{x+2}{\sqrt{6}}\right)+c$

(B) $\tan^{-1}\left(\dfrac{x+2}{\sqrt{6}}\right)+c$

(C) $\dfrac{1}{2\sqrt{6}}\ln\left|\dfrac{x+2-\sqrt{6}}{x+2+\sqrt{6}}\right|+c$

(D) $\ln\left|\dfrac{x+2-\sqrt{6}}{x+2+\sqrt{6}}\right|+c$

10. Which of the following is equal to $\displaystyle\int_0^{2a} f(x)\,dx$? **1**

(A) $\displaystyle\int_0^a \left(f(x)-f(2a-x)\right)\,dx$

(B) $\displaystyle\int_0^a \left(f(x)+f(2a-x)\right)\,dx$

(C) $\displaystyle 2\int_0^a f(x-a)\,dx$

(D) $\displaystyle\int_0^a \frac{1}{2}f(2x)\,dx$

Question 11

(b) Use integration by parts to evaluate $\displaystyle\int_1^e x\ln x\,dx$ **3**

Question 13

(d) i. Show that for any integer n, $e^{in\theta}+e^{-in\theta}=2\cos(n\theta)$. **1**

ii. By expanding $\left(e^{in\theta}+e^{-in\theta}\right)^4$, show that **3**

$$\cos^4\theta=\frac{1}{8}\Big(\cos(4\theta)+4\cos(2\theta)+3\Big)$$

iii. Hence, or otherwise, find $\displaystyle\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta$ **2**

Question 16

(b) Let $I_n=\displaystyle\int_0^{\frac{\pi}{2}}\sin^{2n+1}(2\theta)\,d\theta$, $n=0,1,\cdots$

i. Prove that $I_n=\dfrac{2n}{2n+1}I_{n-1}$, $n\geq 1$. **3**

ii. Deduce that $I_n=\dfrac{2^{2n}\,(n!)^2}{(2n+1)!}$ **3**

$$\text{Let } J_n=\int_0^1 x^n(1-x)^n\,dx,\ n=0,1,2,\cdots$$

(b) i. Using the result of part (ii) or otherwise, show that $J_n=\dfrac{(n!)^2}{(2n+1)!}$. **3**

ii. Prove that $(2^n n!)^2\leq\big(2n+1\big)!$. **2**

A.27 2021 HSC

2. Which expression is equal to $\int x^5 e^{7x}\,dx$? **1**

(A) $\frac{1}{7}x^5e^{7x} - \frac{5}{7}\int x^4e^{7x}\,dx$

(B) $\frac{1}{7}x^5e^{7x} - \frac{5}{7}\int x^5e^{7x}\,dx$

(C) $\frac{1}{7}x^4e^{7x} - \frac{5}{7}\int x^4e^{7x}\,dx$

(D) $\frac{1}{7}x^4e^{7x} - \frac{5}{7}\int x^5e^{7x}\,dx$

Question 11

(f) Express $\dfrac{3x^2-5}{(x-2)\left(x^2+x+1\right)}$ as a sum of partial fractions over $\mathbb{R}$. **3**

Question 12

(a) Find $\displaystyle\int \frac{2x+3}{x^2+2x+2}\,dx$. **3**

Question 13

(b) Use an appropriate substitution to evaluate $\int_{\sqrt{10}}^{\sqrt{13}} x^3\sqrt{x^2-9}\,dx$ **3**

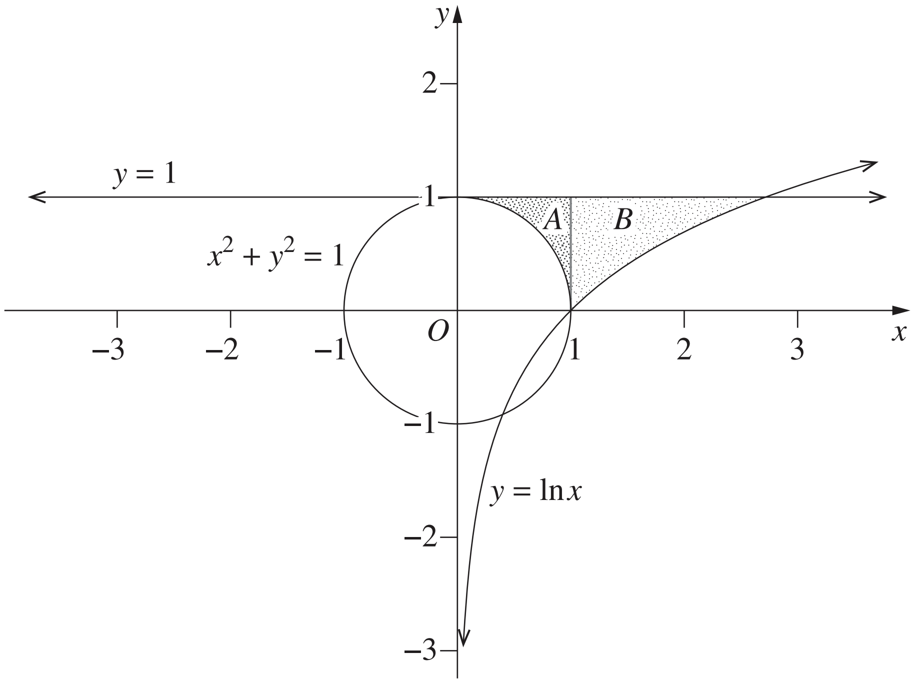
(c) i. The integral I_n is defined by $I_n = \int_1^e (\ln x)^n \; dx$ for integers $n \geq 0$. **2**

Show that $I_n = e - nI_{n-1}$ for $n \geq 1$.

ii. The diagram shows two regions. **4**

Region A is bounded by $y = 1$ and $x^2 + y^2 = 1$ between $x = 0$ and $x = 1$.

Region B is bounded by $y = 1$ and $y = \ln x$ between $x = 1$ and $x = e$.



The volume of the solid created when the region between the curve $y = f(x)$ and the x axis, between $x = a$ and $x = b$, is rotated about the x axis is given by $V = \pi \int_a^b [f(x)]^2 \; dx$.

The volume of the solid of revolution formed when region A is rotated about the x axis is V_A.

The volume of the solid of revolution formed when region B is rotated about the x axis is V_B.

Using part (i), or otherwise, show that the ratio $V_A : V_B$ is $1 : 3$.

Question 14

(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{3 + 5\cos x} \, dx$. **4**

Section B

Coroneos (2004) "100"

1. $\int \frac{x}{x^2+4}\,dx$

2. $\int \frac{x}{\sqrt{x^2+4}}\,dx$

3. $\int \frac{5x+2}{x^2-4}\,dx$

4. $\int \sin x \cos^3 x\,dx$

5. $\int \sin x \sec^3 x\,dx$

6. $\int \cos^2 \frac{x}{2}\,dx$

7. $\int x \sin x\,dx$

8. $\int x \sec^2 2x\,dx$

9. $\int \tan^{-1} 2x\,dx$

10. $\int \frac{x^3}{x^2+1}\,dx$

11. $\int \frac{x}{(x+2)(x+4)}\,dx$

12. $\int \frac{(x-1)(x+1)}{(x-2)(x-3)}\,dx$

13. $\int \frac{2x-1}{x^2+2x+3}\,dx$

14. $\int \frac{x^3}{2x-1}\,dx$

15. $\int \frac{1+x}{\sqrt{1-x-x^2}}\,dx$

16. $\int \frac{1}{x^2\left(1-x^2\right)^{\frac{1}{2}}}\,dx$

17. $\int \frac{1}{x\sqrt{a^2+x^2}}\,dx$

18. $\int \frac{1}{x\sqrt{a^2-x^2}}\,dx$

19. $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx$

20. $\int \frac{x}{\sqrt{x}+1}\,dx$

21. $\int \frac{\cos^{-1} x}{\sqrt{1-x^2}}\,dx$

22. $\int \sqrt{\frac{x+1}{x-1}}\,dx$

23. $\int \frac{1}{x\left(\log_e x\right)^3}\,dx$

24. $\int \sec^4 3x\,dx$

25. $\int \frac{1}{x^2(1-x)}\,dx$

26. $\int \frac{1}{x^2\left(1+x^2\right)}\,dx$

27. $\int \frac{1}{\left(1+x^2\right)^2}\,dx$

28. $\int \tan^3 x\,dx$

29. $\int \frac{1}{5+3\cos x}\,dx$

30. $\int \frac{1}{3+5\cos x}\,dx$

31. $\int \frac{\sin x}{5+3\cos x}\,dx$

32. $\int \frac{1}{1+\cos^2 x}\,dx$

33. $\int \frac{1}{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}\,dx$

34. $\int x^2 \sin x\,dx$

35. $\int \frac{x^2}{(x-1)(x-2)(x-3)}dx$

36. $\int \frac{e^x}{e^x-1}\,dx$

37. $\int \frac{1}{3\sin^2 x + 5\cos^2 x}\,dx$

38. $\int x^3 e^{5x^4-7}\,dx$

39. $\int x^5 \log_e x\,dx$

40. $\int \frac{3x+2}{x(x+1)^3}\,dx$

41. $\int \log\left(x^3\right)\,dx$

42. $\int \frac{1}{e^x+e^{-x}}\,dx$

43. $\int \left(5x^3+7x-1\right)^{\frac{3}{2}} \times \left(15x^2+7\right)\,dx$

44. $\int \frac{1}{\left(x^2+4\right)\left(x^2+1\right)}\,dx$

45. $\int \left(x^2+x+1\right)^{-1}\,dx$

46. $\int e^x \sin 2x \, dx$

47. $\int \left(x^2 + x - 1\right)^{-1} \, dx$

48. $\int \left(x^2 - x\right)^{-\frac{1}{2}} \, dx$

49. $\int \frac{1 - 2x}{3 + x} \, dx$

50. $\int x^3 \left(4 + x^2\right)^{-\frac{1}{2}} \, dx$

51. $\int \frac{\sin 2x}{3\cos^2 x + 4\sin^2 x} \, dx$

52. $\int \frac{x^2}{1 - x^4} \, dx$

53. $\int \frac{1}{\sin x \cos x} \, dx$

54. $\int \log_e \sqrt{x - 1} \, dx$

55. $\int \frac{1}{e^x - 1} \, dx$

56. $\int \frac{\sec^2 x}{\tan^2 x - 3\tan x + 2} \, dx$

57. $\int \frac{x + 1}{(x^2 - 3x + 2)^{\frac{1}{2}}} \, dx$

58. $\int \sin 2x \cos x \, dx$

59. $\int \frac{x}{1 + x^3} \, dx$

60. $\int x \tan^{-1} x \, dx$

61. $\int \left(1 + 3x + 2x^2\right)^{-1} \, dx$

62. $\int \left(9 - x^2\right)^{\frac{1}{2}} \, dx$

63. $\int \left(9 + x^2\right)^{\frac{1}{2}} \, dx$

64. $\int x \left(9 + x^2\right)^{\frac{1}{2}} \, dx$

65. $\int \sec^2 x \tan^3 x \, dx$

66. $\int x^2 e^{-x} \, dx$

67. $\int x e^{x^2} \, dx$

68. $\int \sin x \tan x \, dx$

69. $\int \sin^4 x \cos^3 x \, dx$

70. $\int \frac{x^3 + 1}{x^3 - x} \, dx$

71. $\int \log_e \left(x + \sqrt{x^2 - 1}\right) \, dx$

72. $\int \frac{1}{(x + 1)^{\frac{1}{2}} + (x + 1)} \, dx$

73. $\int_0^4 \frac{x}{\sqrt{x + 4}} \, dx$

74. $\int_1^2 \frac{1}{x\,(1 + x^2)} \, dx$

75. $\int_1^2 \frac{\log_e x}{x} \, dx$

76. $\int_0^1 \cos^{-1} x \, dx$

77. $\int_1^2 \frac{x + 1}{\sqrt{-2 + 3x - x^2}} \, dx$

78. $\int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x + 2\sin^2 x} \, dx$

79. $\int_0^1 x\sqrt{1 - x^2} \, dx$

80. $\int_2^4 x \log_e x \, dx$

81. $\int_1^2 \frac{1}{x^2 + 5x + 4} \, dx$

82. $\int_0^{\frac{\pi}{2}} \left(1 + \frac{1}{2}\sin x\right)^{-1} \, dx$

83. $\int_0^1 x^2 e^{-x} \, dx$

84. $\int_0^1 \frac{7 + x}{1 + x + x^2 + x^3} \, dx$

85. $\int_0^1 \frac{e^{-2x}}{e^{-x} + 1} \, dx$

86. $\int_0^{\frac{a}{2}} \frac{y}{a - y} \, dy$

87. $\int_0^a \frac{(a - x)^2}{a^2 + x^2} \, dx$

88. $\int_0^1 \frac{x + 3}{(x + 2)(x + 1)^2} \, dx$

89. $\int_0^1 \frac{x^2}{x^6 + 1} \, dx$

90. $\int_0^\pi \cos^2 mx \, dx, \; m \in \mathbb{Z}$

91. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \sin 2x \, dx$

92. $\int_0^{\frac{a}{2}} x^2 \sqrt{a^2 - x^2} \, dx$

93. $\int_0^{\frac{\pi}{4}} \sec^2 x \tan x \, dx$

94. $\int_0^1 (x+2) \left(x^2 + 4x + 5\right)^{\frac{1}{2}} \, dx$

95. $\int_1^2 x \left(\log_e x\right)^2 \, dx$

96. $\int_3^4 \frac{x^2 + 4}{x^2 - 1} \, dx$

97. $\int_1^4 \frac{x^2 + 4}{x(x + 2)} \, dx$

98. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{5 - 3\sin x} \, dx$

99. $\int_0^1 \frac{1}{(4 - x^2)^{\frac{3}{2}}} \, dx$

100. $\int_0^{\frac{\pi}{2}} 2\sin\theta\cos\theta \left(3\sin\theta - 4\sin^3\theta\right) d\theta$

Answers

1. $\frac{1}{2}\ln\left(x^2+4\right)$ **2.** $\sqrt{x^2+4}$ **3.** $3\ln(x-2)+2\ln(x+2)$ **4.** $-\frac{1}{4}\cos^4 x$ **5.** $\frac{1}{2}\sec^2 x$ **6.** $\frac{1}{2}(x+\sin x)$ **7.** $-x\cos x+\sin x$ **8.** $\frac{1}{2}x\tan 2x+\frac{1}{4}\ln(\cos 2x)$ **9.** $x\tan^{-1}2x-\frac{1}{4}\ln\left(1+4x^2\right)$ **10.** $\frac{1}{2}x^2-\frac{1}{2}\ln\left(1+x^2\right)$ **11.** $2\ln(x+4)-\ln(x+2)$ **12.** $x-3\ln(x-2)+8\ln(x-3)$ **13.** $\ln\left(x^2+2x+3\right)-\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$ **14.** $\frac{1}{6}x^3+\frac{1}{8}x^2+\frac{1}{8}x+\frac{1}{16}\ln(2x-1)$ **15.** $\frac{1}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)-\sqrt{1-x-x^2}$ **16.** $-\frac{\sqrt{1-x^2}}{x}$ **17.** $-\frac{1}{a}\ln\left(\frac{\sqrt{a^2+x^2}+a}{x}\right)$ or $-\frac{1}{a}\ln\left(\frac{x}{\sqrt{a^2+x^2}-a}\right)$ **18.** $-\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$ or $-\frac{1}{a}\ln\left(\frac{x}{a-\sqrt{a^2-x^2}}\right)$ **19.** $\frac{1}{a}\sec^{-1}\frac{x}{a}$ **20.** $\frac{3}{2}x^{\frac{3}{2}}-x+2x^{\frac{1}{2}}-2\log\left(1+x^{\frac{1}{2}}\right)$ **21.** $-\frac{1}{2}\left(\cos^{-1}x\right)^2$ **22.** $\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)$ **23.** $-\frac{1}{2(\ln x)^2}$ **24.** $\frac{1}{3}\tan 3x+\frac{1}{9}\tan^3 3x$ **25.** $\ln x-\frac{1}{x}-\ln(1-x)$ **26.** $-\frac{1}{x}-\tan^{-1}x$ **27.** $\frac{1}{2}\tan^{-1}x+\frac{x}{2(1+x^2)}$ **28.** $\frac{1}{2}\tan^2 x+\ln\cos x$ **29.** $\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)$ **30.** $\frac{1}{4}\ln\left(\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right)$ **31.** $-\frac{1}{3}\ln(5+3\cos x)$ **32.** $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)$ **33.** $\ln(\sec x+\tan x)$ **34.** $-x^2\cos x+2x\sin x+2\cos x$ **35.** $\frac{1}{2}\ln(x-1)-4\ln(x-2)+\frac{9}{2}\ln(x-3)$ **36.** $\ln\left(e^x-1\right)$ **37.** $\frac{1}{\sqrt{15}}\tan^{-1}\left(\sqrt{\frac{3}{5}}\tan x\right)$ **38.** $\frac{1}{20}e^{5x^4-7}$ **39.** $\frac{1}{6}x^6\ln x-\frac{1}{36}x^6$ **40.** $2\ln x-2\ln(x+1)+\frac{2}{x+1}-\frac{1}{2(x+1)^2}$ **41.** $3(x\ln x-x)$ **42.** $\tan^{-1}\left(e^x\right)$ **43.** $\frac{2}{5}\left(5x^3+7x-1\right)^{\frac{5}{2}}$ **44.** $\frac{1}{3}\left(\tan^{-1}x-\frac{1}{2}\tan^{-1}\frac{x}{2}\right)$ **45.** $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$ **46.** $\frac{e^x}{5}(\sin 2x-2\cos 2x)$ **47.** $\frac{1}{\sqrt{5}}\ln\left(\frac{2x+1-\sqrt{5}}{2x+1+\sqrt{5}}\right)$ **48.** $\ln\left(\left(x-\frac{1}{2}\right)+\sqrt{x^2-x}\right)$ **49.** $-2x+7\ln(3+x)$ **50.** $\frac{1}{3}\left(x^2-8\right)\sqrt{4+x^2}$ **51.** $\ln\left(3+\sin^2 x\right)$ **52.** $\frac{1}{4}\ln(1+x)-\frac{1}{4}\ln(1-x)-\frac{1}{2}\tan^{-1}x$ **53.** $\ln\tan x$ or $\ln(\operatorname{cosec}2x+\cot 2x)$ **54.** $\frac{1}{2}(x-1)\ln(x-1)-\frac{1}{2}x$ **55.** $\ln\left(e^x-1\right)-x$ **56.** $\ln\left(\frac{\tan x-2}{\tan x-1}\right)$ **57.** $\sqrt{x^2-3x+2}+\frac{5}{2}\ln\left(x-\frac{3}{2}+\sqrt{x^2-3x+2}\right)$ **58.** $-\frac{2}{3}\cos^3 x$ **59.** $\frac{1}{6}\ln\left(1-x+x^2\right)-\frac{1}{3}\ln(1+x)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$ **60.** $\frac{1}{2}\left(x^2\tan^{-1}x+\tan^{-1}x-x\right)$ **61.** $\ln\frac{1+2x}{1+x}$ **62.** $\frac{1}{2}\left(x\sqrt{9-x^2}+9\sin^{-1}\frac{x}{3}\right)$
63. $\frac{1}{2}\left(x\sqrt{9+x^2}+9\ln\left(x+\sqrt{9+x^2}\right)\right)$ **64.** $\frac{1}{3}\left(9+x^2\right)^{\frac{3}{2}}$ **65.** $\frac{1}{4}\tan^4 x$ **66.** $-e^{-x}\left(x^2+2x+2\right)$ **67.** $\frac{1}{2}e^{x^2}$ **68.** $\ln(\sec x+\tan x)-\sin x$ **69.** $\frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x$ **70.** $x+\ln(x-1)-\ln x$ **71.** $x\ln\left(x+\sqrt{x^2-1}\right)-\sqrt{x^2-1}$ **72.** $2\ln\left(1+\sqrt{1+x}\right)$ **73.** $\frac{16}{3}\left(2-\sqrt{2}\right)$ **74.** $\frac{1}{2}\ln\frac{8}{5}$ **75.** $\frac{1}{2}(\ln 2)^2$ **76.** 1 **77.** $\frac{5\pi}{2}$ **78.** $\frac{\pi\sqrt{2}}{4}$ **79.** $\frac{1}{3}$ **80.** $14\ln 2-3$ **81.** $\frac{1}{3}\ln\frac{5}{4}$ **82.** $\frac{2\pi}{3\sqrt{3}}$ **83.** $2-\frac{5}{e}$ **84.** $\frac{3}{2}\ln 2+\pi$ **85.** $\ln\left(\frac{e+1}{2e}\right)-\frac{1}{e}+1$ **86.** $\frac{a}{2}(\ln 4-1)$ **87.** $a(1-\ln 2)$ **88.** $1+\ln\frac{3}{4}$ **89.** $\frac{\pi}{12}$ **90.** $\frac{\pi}{2}$ **91.** $\frac{1}{4}(\pi-1)$ **92.** $\frac{(4\pi-3\sqrt{3})a^4}{192}$ **93.** $\frac{1}{2}$ **94.** $\frac{5\sqrt{5}}{3}\left(2\sqrt{2}-1\right)$ **95.** $2(\ln 2)^2-2\ln 2+\frac{3}{4}$ **96.** $1+\frac{5}{2}\ln\frac{6}{5}$ **97.** 3 **98.** $\frac{1}{3}\ln\frac{5}{2}$ **99.** $\frac{1}{4\sqrt{3}}$ **100.** $\frac{2}{5}$

Important note

Worked solutions are available on YouTube by Mr Jason Wang.

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$l = \frac{\theta}{360} \times 2\pi r$

Area

$A = \frac{\theta}{360} \times \pi r^2$

$A = \frac{h}{2}(a + b)$

Surface area

$A = 2\pi r^2 + 2\pi rh$

$A = 4\pi r^2$

Volume

$V = \frac{1}{3}Ah$

$V = \frac{4}{3}\pi r^3$

Functions

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For $ax^3 + bx^2 + cx + d = 0$:

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$(x - h)^2 + (y - k)^2 = r^2$

Financial Mathematics

$A = P(1 + r)^n$

Sequences and series

$T_n = a + (n - 1)d$

$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$

$T_n = ar^{n-1}$

$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$

$S = \frac{a}{1 - r}, \ |r| < 1$

Logarithmic and Exponential Functions

$\log_a a^x = x = a^{\log_a x}$

$\log_a x = \frac{\log_b x}{\log_b a}$

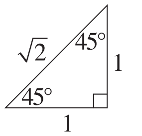
$a^x = e^{x \ln a}$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

$A = \dfrac{1}{2}ab\sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

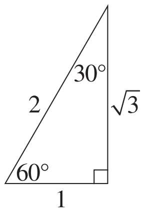


$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

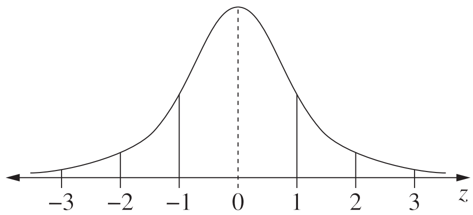
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x} p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

References

Arnold, D., & Arnold, G. (2000). *Cambridge Mathematics 4 Unit* (2nd ed.). Cambridge University Press.

Coroneos, J. (2004). *Revised 4 Unit Course for Mathematics Extension 2*. Coroneos Publishing.

Lee, T. (2006). *Advanced Mathematics: A complete HSC Mathematics Extension 2 Course* (2nd ed.). Terry Lee Enterprise.

Patel, S. K. (1990). *Excel 4 Unit Maths*. Pascal Press.

Patel, S. K. (2004). *Maths Extension 2* (2nd ed.). Pascal Press.

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019). *CambridgeMATHS Stage 6 Mathematics Extension 1 Year 12* (1st ed.). Cambridge Education.

Sadler, D., & Ward, D. (2019). *CambridgeMATHS Stage 6 Mathematics Extension 2* (1st ed.). Cambridge Education.